Forcing large tight components in 3-graphs

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Abstract

Any n-vertex 3-graph with minimum codegree at least $\lfloor n/3 \rfloor$ must have a spanning tight component, but immediately below this threshold it is possible for no tight component to span more than $\lceil 2n/3 \rceil$ vertices. Motivated by this observation, we ask which co-degree forces a spanning tight component of any given size. The corresponding function seems to have infinitely many discontinuities towards the origin, making it hard to analyse, but we provide upper and lower bounds, which asymptotically converge as the function nears the origin.

1 Introduction

This paper addresses the extremal question of which minimum codegree forces a tight component containing at least a certain proportion of the vertices of a 3-uniform hypergraph.

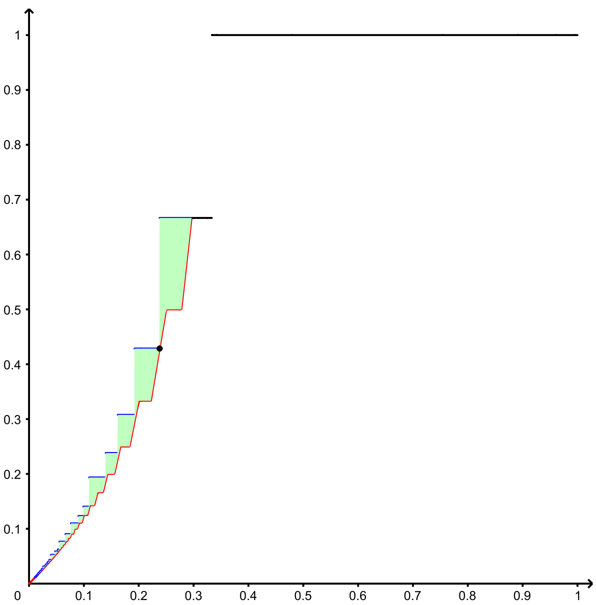

Figure 1: Our upper (blue, if colour is shown) and lower (red) bounds on $f_3(x)$. These bounds coincide for $x \in \left\{\frac{5}{21}\right\} \cup \left[\frac{8}{27}, 1\right]$.

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A k-graph is a hypergraph $\mathcal{H}$, every edge of which consists of k vertices; we will concentrate on the case $k = 3$. We say that two edges of $\mathcal{H}$ *touch* if they share $k-1$ vertices. The *tight components* of $\mathcal{H}$ are the equivalence classes of edges under the transitive closure of this relation. Equivalently, two edges are in the same tight component if and only if the hypergraph contains a tight walk between them, that is, a chain of edges where each edge has the maximum possible overlap with the next. A special case of tight walks are tight paths, which were introduced by Katona and Kierstead [9] and are one of the most natural generalisations of paths to hypergraphs. Tight paths or cycles feature in many works on hypergraphs, for example [12, 7, 1]. Consequently, tight components are a natural generalisation of the notion of the components of a graph. The property of *hypergraph connectivity*, that is, having a tight component covering all $(k-1)$-tuples, was studied by Kahle and Pittel [8], and is closely related to the earlier definition of homological connectivity by Linial and Meshulam [11]. The emergence and size of the giant tight component was analysed by Cooley, Kang and Person [4] and Cooley, Kang and Koch [3]. Our interest in tight components was motivated by our work with Narayanan on the minimum codegree required for a spanning surface [6], in which the absence of a spanning tight component is the main obstacle.

The *codegree* of a $(k-1)$-tuple of vertices is the number of edges containing that tuple. We write $\delta_{k-1}(\mathcal{H})$ for the minimum codegree over all $(k-1)$-tuples of $\mathcal{H}$.

Definition 1. *We define a function $f_k(x) : [0,1] \to [0,1]$ by letting $f_k(x)$ be the largest real number such that every n-vertex k-graph with minimum codegree at least $xn - O(1)$ has a tight component meeting at least $f_k(x)n$ of its vertices. (Omitting the $-O(1)$ term in this definition changes f_r only slightly: from left-continuous to right-continuous.)*

The function f_2 is easy to analyse. Any n-vertex graph G with $\delta(G) \geq \lfloor n/m \rfloor$ can have at most $m-1$ components, so one of them meets at least $\lceil n/(m-1) \rceil$ vertices. Conversely, if $k < \lfloor n/m \rfloor$ there is a graph with m components meeting $\lfloor n/m \rfloor$ or $\lceil n/m \rceil$ vertices which has minimum degree at least k. Thus $f_2(x) = \frac{1}{\lfloor 1/x \rfloor}$.

In this paper we analyse f_3; we provide upper and lower bounds as shown in Figure 1. These bounds become asymptotically tight as $x \to 0$. Our upper bounds are based on the existence of finite projective planes of certain orders.

It turns out that f_3 is discontinuous at $x = 1/3$. We conjecture that it has infinitely many discontinuities. The higher functions $f_k, k > 3$ might be much harder to analyse.

Problem 1. *Provide asymptotic formulae for f_k, when $k > 3$.*

2 Spanning tight components

First we note that a minimum codegree of $\lfloor n/3 \rfloor$ is sufficient to force a spanning tight component, as shown below. This fact was pointed out to us by Richard Mycroft (private communication). This bound is best possible as proved by the following example. Consider the 3-graph whose vertices are partitioned into three sets V_0, V_1, V_2, of as equal sizes as possible, with all edges consisting of three vertices in V_i or of two vertices in V_i and one in V_{i+1}, for some $i \in \mathbb{Z}_3$. Each tight component only meets vertices in two parts, so is far from spanning, yet the minimum codegree is $\lfloor n/3 \rfloor - 1$.

Proposition 2 (R. Mycroft (private communication))**.** *Any 3-graph $\mathcal{H}$ on n vertices with minimum codegree at least $\lfloor n/3 \rfloor$ has at most two tight components.*

Proof. Suppose not. Let K_n denote the complete graph on the vertices of $\mathcal{H}$. Colour the hyperedges of $\mathcal{H}$ according to the tight component they are in, and give each edge $e = uv$ of K_n the colour of those hyperedges of $\mathcal{H}$ which contain $\{u, v\}$.

First, we show that no triangle of K_n has more than two colours. If xy, yz and zx are different colours then consider the sets of vertices A, B, C which can be used to extend xy, yz, zx respectively to hyperedges. Every vertex in A has two edges of the first colour to $\{x, y, z\}$, etc., so these sets are disjoint from each other and $\{x, y, z\}$. But each has size at least $\lfloor n/3 \rfloor$, so we have $n \geq 3 + 3\lfloor n/3 \rfloor$, a contradiction.

Second, we show that no vertex meets three colours. If v does, say red, green and blue, let R (respectively, G or B) be the sets of vertices connected to v by red (respectively, green or blue) edges.

These are disjoint, but if vx is any red edge then there are at least $\lfloor n/3 \rfloor$ vertices which extend it to a red hyperedge, so $|R| \geq 1 + \lfloor n/3 \rfloor$. The same applies to G and B, so $|R| + |G| + |B| > n$, contradiction.

Now consider any vertex v which meets edges of two colours (this trivially exists), say red and blue. Let R be the set of all vertices with red edges to v, and define B similarly. R and B partition $V - v$. By assumption, a third colour, green, is used somewhere; it cannot be between R and B as there are no 3-coloured triangles, so it is within R, say. If xy is such an edge then vx extends to $\lfloor n/3 \rfloor$ red hyperedges and xy extends to $\lfloor n/3 \rfloor$ green hyperedges. Each of the vertices which extends one of these two is in R (if vxz is a red hyperedge then vz is red; if xyz is green then $z \notin B$ since that would create a 3-coloured triangle). So $|R| \geq 2 + 2\lfloor n/3 \rfloor$ and as before $|B| \geq 1 + \lfloor n/3 \rfloor$, giving a contradiction. □

Corollary 3 (R. Mycroft (private communication))**.** *Any 3-graph $\mathcal{H}$ on n vertices with $\delta_2(\mathcal{H}) \geq \lfloor n/3 \rfloor$ has a spanning tight component.*

Proof. If the first tight component does not meet some vertex x then for each other vertex y, the edges containing x and y must belong to the other tight component, which therefore meets all vertices. □

The example given above shows that reducing the minimum codegree condition even by 1 allows hypergraphs where no tight component meets more than $\lceil 2n/3 \rceil$ vertices. This is the motivation for Question 1: we have shown that $f_3(x) = 1$ for all $x > 1/3$, but $f_3(1/3) \leq 2/3$.

We will show that a minimum codegree of $n/r - O(1)$ implies that some tight component meets at least $n/(r-2) - O(1)$ vertices for each integer $r \geq 3$. We also show that this is almost best possible: for infinitely many values of r there are hypergraphs with minimum codegree $(1/r - O(r^{-3}))n$ in which no tight component meets more than $(1/(r-2) - O(r^{-3}))n$ vertices.

3 Upper bounds

In this section we give a construction based on finite projective planes. A finite projective plane of order s is an arrangement of points and lines such that each point lies on $s+1$ lines, each line contains $s+1$ points, each pair of points is contained in a unique line, and each pair of lines meet in a unique point. Such a structure is known to exist whenever s is a prime power. Bruck and Ryser [2] proved that if $s \equiv 1 \pmod 4$ or $s \equiv 2 \pmod 4$, and s is not the sum of two squares, then no projective plane exists. The existence of a projective plane of order 10 was ruled out by extensive computer analysis, completed by Lam, Thiel and Swiercz [10], but for every other value of s which is neither a prime power nor ruled out by the Bruck–Ryser result, it is an open question. We will consider a projection as a hypergraph, where the vertices are the points and the edges are the lines.

Let $\mathrm{tc}(\mathcal{H})$ denote the number of vertices of the largest tight component of the 3-graph $\mathcal{H}$.

Theorem 4. *For each $r \geq 3$ for which a projective plane of order $r-2$ exists, and any n, there exists an n-vertex 3-graph $\mathcal{H}$ satisfying*

$$\delta_2(\mathcal{H}) = \left(\frac{r - 3 + \frac{2}{r-1}}{r^2 - 3r + 3}\right)n - O(1) \quad \textit{and}$$
$$\mathrm{tc}(\mathcal{H}) = \left(\frac{r-1}{r^2 - 3r + 3}\right)n + O(1)\,.$$

Remark. In fact, provided $r^2 - 3r + 3 \mid n$, we do not need the $+O(1)$ term in the latter expression.

Proof. Let $\mathcal{P}_{r-2}$ be a finite projective plane of order $r-2$; this is an $(r-1)$-uniform hypergraph with $r^2 - 3r + 3$ vertices $v_1, \ldots, v_{r^2-3r+3}$ and $r^2 - 3r + 3$ edges, with each vertex having degree $r-1$ and each pair of vertices contained in exactly one edge. Associate each edge with a different colour.

Colour the complete graph K_n on n vertices as follows. Divide the vertices as evenly as possible into $r^2 - 3r + 3$ classes $C_1, \ldots, C_{r^2-3r+3}$. For each class C_i, using the $r-1$ colours corresponding to the edges of $\mathcal{P}_{r-2}$ meeting v_i, colour the edges within C_i such that for each vertex the numbers of incident edges of each colour are as equal as possible. Colour each edge between two classes C_i and C_j, where $i \neq j$, according to the unique edge of $\mathcal{P}_{r-2}$ which contains v_i and v_j. Figure 2 shows such a colouring for $r = 4$.

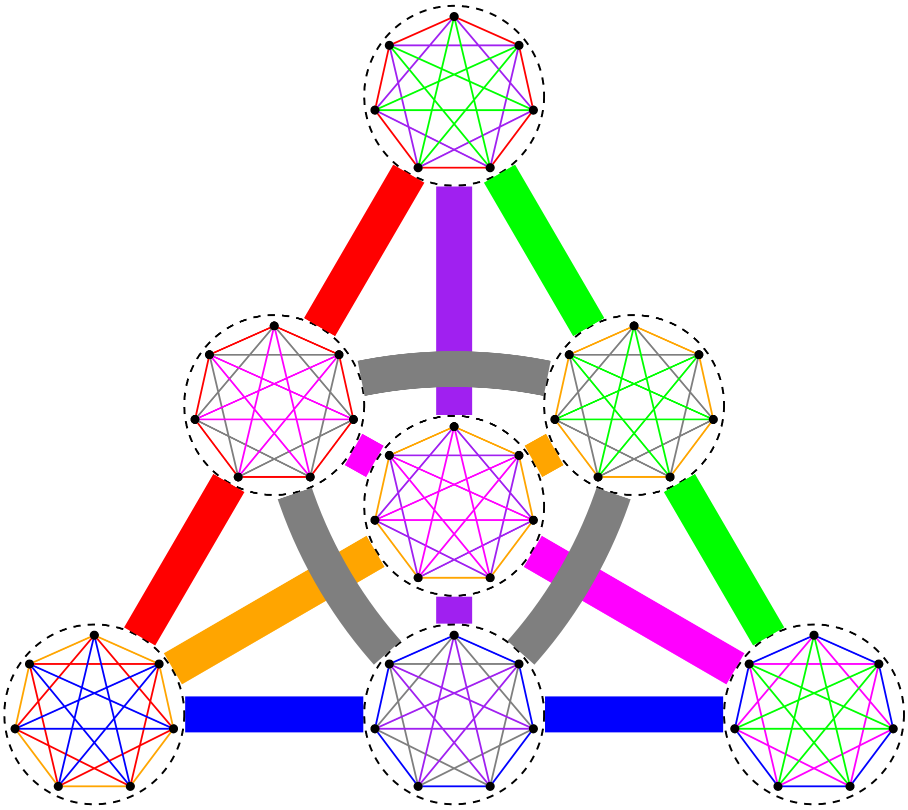

Figure 2: A construction based on $\mathcal{P}_2$.

Now define a hypergraph $\mathcal{H}$ on the vertex set of K_n, whose edges are the monochromatic triangles of this colouring of K_n. If we give edges of $\mathcal{H}$ the same colour as the corresponding triangle, each tight component is monochromatic, and each colour touches $r-1$ classes C_i, so for large n each tight component has at most as many vertices as the $r-1$ largest classes. It is easy to see that each colour corresponds to at most one tight component, so we have

$$\mathrm{tc}(\mathcal{H}) = \frac{r-1}{r^2-3r+3}n + O(1)\,.$$

Fix a pair of vertices $x, y \in V(\mathcal{H})$, and let e be the edge of $\mathcal{P}_{r-2}$ corresponding to the colour of xy. The *degree* $d_{\mathcal{H}}(x,y)$ is the number of hyperedges of $\mathcal{H}$ containing both x, y. If x and y are in the same class, all vertices in the other $r-2$ classes corresponding to vertices of e all form monochromatic triangles with x, y, so we have

$$d_{\mathcal{H}}(x,y) \geq \frac{r-2}{r^2-3r+3}n + O(1)\,.$$

If $x \in C_i$ and $y \in C_j$ with $i \neq j$, then all vertices in the other $r-3$ classes corresponding to vertices of e form monochromatic triangles with x, y, as do the vertices in C_i with an appropriately coloured edge to x, and those in C_j with an appropriately coloured edge to y. In total, we have

$$d_{\mathcal{H}}(x,y) = \frac{r-2+\frac{2}{r-1}}{r^2-3r+3}n + O(1)\,.$$

Thus we have

$$\delta_2(\mathcal{H}) \geq \min\left(\frac{r-2}{r^2-3r+3}, \frac{r-3+\frac{2}{r-1}}{r^2-3r+3}\right)n + O(1)\,.$$

For $r = 3$ the two bounds coincide, and for $r > 3$ the latter is smaller, giving the required equality. $\square$

Remark. Writing $s = \frac{r^2-3r+3}{r-3+\frac{2}{r-1}}$, we have $\frac{r^2-3r+3}{r-1} = s-2+O(r^{-3}) = s-2+O(s^{-3})$. Thus we have, for infinitely many values of s, hypergraphs with minimum codegree $n/s - O(1)$ and no tight component meeting more than $(1/(s-2)+O(s^{-3}))n$ vertices.

4 Lower bounds

Next we give a lower bound which is close to the upper bound of the previous section for large r. We will use the following result of Füredi on fractional matchings in hypergraphs [5]. A *matching* in a hypergraph $\mathcal{H}$ is a set of disjoint edges, and the matching number $\nu(\mathcal{H})$ is the maximum size of a matching in $\mathcal{H}$. A *fractional matching* is a weight function $w : E(\mathcal{H}) \to [0,1]$ such that $\sum_{e\ni v} w(e) \le 1$ for each $v \in V(\mathcal{H})$, and the *fractional matching number* $\nu^*(\mathcal{H})$ is the maximum of $\sum_{e\in E(\mathcal{H})} w(e)$ over all fractional matchings.

Theorem 5 (Füredi [5])**.** *Let $\mathcal{H}$ be a hypergraph with edges of size at most k which does not contain $p+1$ vertex-disjoint projective planes of order $k-1$, for some $k \ge 3$ and $p \ge 0$. Then $\nu^*(\mathcal{H}) \le (k-1)\nu(\mathcal{H})+p/k$.*

We write $\Delta_1(\mathcal{H})$ for the maximum vertex degree of a hypergraph $\mathcal{H}$.

Corollary 6. *If $\mathcal{H}$ is a k-uniform multi-hypergraph for which any two edges intersect then*

$$\Delta_1(\mathcal{H}) \ge \frac{e(\mathcal{H})}{k-1+p/k},$$

where $p = 1$ if a projective plane of order $k-1$ exists, and $p = 0$ otherwise. Further, if $k \ge 3$ and $\Delta_1(\mathcal{H}) > \frac{e(\mathcal{H})}{k-1}$ then the underlying simple graph is a projective plane

Proof. If $k = 2$ then either there is a vertex in every edge or $\mathcal{H}$ has only three vertices; in the latter case the average degree is $2e(\mathcal{H})/3$. If $k \ge 3$ we may apply Theorem 5 to the underlying simple hypergraph $\mathcal{H}'$. Since $\mathcal{H}'$ is intersecting, $\nu(\mathcal{H}') = 1$, and so $\nu^*(\mathcal{H}) \le k-1+p/k$. Let $w(e)$ be the number of copies of e in the multi-hypergraph $\mathcal{H}$, divided by $\Delta_1(\mathcal{H})$. Clearly, for each $v \in V(\mathcal{H})$,

$$\sum_{e\ni v} w(e) = \frac{d_{\mathcal{H}}(v)}{\Delta_1(\mathcal{H})} \le 1,$$

so w is a fractional matching for $\mathcal{H}'$. Thus,

$$k-1+p/k \ge \sum_{e\in E(\mathcal{H}')} w(e) = \frac{e(\mathcal{H})}{\Delta_1(\mathcal{H})},$$

giving the required bound. If $k \ge 3$ and $\nu^*(\mathcal{H}') > k-1$ then $\mathcal{H}'$ contains a projective plane, and, since it is intersecting, no other edges. □

Theorem 7. *Fix an integer $r \ge 3$. Suppose $\mathcal{H}$ is a 3-uniform hypergraph on n vertices with $\delta_2(\mathcal{H}) \ge (1-\varepsilon)n/r$, where $0 \le \varepsilon < \frac{1}{r+1}$. Then*

$$\mathrm{tc}(\mathcal{H}) \ge \begin{cases} \min\{(1-3\varepsilon), 2/3\}n & \text{if } r = 3 \\ (1-3\varepsilon)\dfrac{n}{r-2} & \text{otherwise.} \end{cases} \tag{1}$$

Proof. Again, we colour the edges of the complete graph on the same vertex set. Give xy the colour of the tight component containing edges of the form xyz. Fix a vertex x. If x meets an edge xy of a particular colour in the graph, there are at least $\delta_2(\mathcal{H})$ edges of the form xyz in $\mathcal{H}$ which are in the corresponding tight component. Thus if x meets an edge of a certain colour, it meets at least $\delta_2(\mathcal{H})$ such edges. Since $\varepsilon < 1/(r+1)$, $\delta_2(\mathcal{H}) > n/(r+1)$, so the number of tight components meeting a vertex x is at most r. We distinguish three cases, as follows.

Case 1. Some vertex meets at most $r-2$ tight components.

In this case, these $r-2$ components must between them cover all the vertices, so at least one must meet at least $n/(r-2)$ vertices.

Case 2. Every vertex meets exactly $r-1$ tight components.

We define an auxiliary multi-hypergraph $\mathcal{F}$ as follows. The vertices of $\mathcal{F}$ correspond to tight components of $\mathcal{H}$. The edges of $\mathcal{F}$ correspond to vertices of $\mathcal{H}$; an edge e_v of $\mathcal{F}$ corresponding to a vertex v of $\mathcal{H}$ contains the $r-1$ vertices of $\mathcal{F}$ corresponding to tight components which meet v. Thus $\mathcal{F}$ is $(r-1)$-uniform and $e(\mathcal{F}) = n$. Any two edges of $\mathcal{F}$ intersect: e_u and e_v both contain the vertex corresponding to the tight component containing edges of the form uvw. If $r = 3$, by Corollary 6, such an $\mathcal{F}$ has a vertex meeting at least $2n/3$ edges, and hence $\mathcal{H}$ has a tight component meeting this many vertices. If $r > 3$, by Corollary 6, either $\mathcal{F}$ has a vertex meeting at least $n/(r-2)$ edges or its underlying simple hypergraph $\mathcal{F}'$ is a projective plane of order $r-2$; in the latter case some vertex of $\mathcal{F}$ meets at least $\frac{1}{r-2+1/(r-1)}n$ edges. Note that

$$\frac{1}{r-2+1/(r-1)} = \left(1 - \frac{1}{r^2-3r+3}\right)\frac{1}{r-2}\,.$$

Claim 1. If $\mathcal{F}'$ is a projective plane of order $r-2$ then $\varepsilon \geq \frac{r-3}{(r-1)(r^2-3r+3)}$.

Proof of Claim 1. If $\mathcal{F}'$ is a projective plane of order $r-2$, then note that its edges give a partition, $C_1, \ldots, C_{r^2-3r+3}$ say, of the vertices of $\mathcal{H}$ so that any two vertices in the same class are in exactly the same tight components. Each class meets $r-1$ tight components, and each pair of classes have a single tight component in common. Colour each pair of vertices of $\mathcal{H}$ according to the tight component the edges containing that pair are in.

We choose a pair of vertices (x, y) as follows: choose uniformly at random between the ordered pairs $(i, j) \in [r^2-3r+3]^2$ which satisfy $i \neq j$, and choose (independently and uniformly at random) $x \in C_i$ and $y \in C_j$. Fix a vertex z and consider $\mathbb{P}(xyz \in E(\mathcal{H}))$. If $z \in C_k$ we have

$$\begin{aligned}\mathbb{P}(xyz \in E(\mathcal{H})) &= \mathbb{P}((xyz \in E(\mathcal{H})) \wedge (k \notin \{i,j\})) + \mathbb{P}((xyz \in E(\mathcal{H})) \wedge (k \in \{i,j\})) \\ &\leq \mathbb{P}(w_i w_j w_k \in E(\mathcal{F}'')) + 2\mathbb{P}((k=i) \wedge (\mathrm{col}(xy) = \mathrm{col}(xz)))\,. \end{aligned} \tag{2}$$

Now

$$\begin{aligned}\mathbb{P}(w_i w_j w_k \in E(\mathcal{F}'')) &= \frac{(r-1)\binom{r-2}{2}}{\binom{r^2-3r+3}{2}} \\ &= \frac{r-3}{r^2-3r+3}\,,\end{aligned} \tag{3}$$

and

$$\begin{aligned}\mathbb{P}((k=i) \wedge (\mathrm{col}(xy) = \mathrm{col}(xz))) &< \frac{1}{r^2-3r+3} \cdot \frac{r-2}{r^2-3r+2} \\ &= \frac{1}{(r-1)(r^2-3r+3)}\,,\end{aligned} \tag{4}$$

since given $i = k$, $\mathrm{col}(xy)$ depends only on j, and for each choice of x (other than $x = z$) there are $r-2$ choices of j which give $\mathrm{col}(xy) = \mathrm{col}(xz)$.

Thus, by (2), (3) and (4), we have

$$\begin{aligned}\delta_2(\mathcal{H}) &\leq \mathbb{E}(d_{\mathcal{H}}(x,y)) \\ &< \frac{r-3+\frac{2}{r-1}}{r^2-3r+3}n \\ &= \left(1 - \frac{r-3}{(r-1)(r^2-3r+3)}\right)\frac{n}{r}\,,\end{aligned}$$

as required. This completes the proof of Claim 1. ■

The desired result follows in this case, since if $\varepsilon \geq \frac{r-3}{(r-1)(r^2-3r+3)}$ and $r \geq 4$ then $3\varepsilon \geq \frac{1}{r^2-3r+3}$.

Case 3. Neither of the above two cases apply.

In this case, some vertex x meets exactly r tight components. Divide the remaining $n-1$ vertices into classes $A_1,\dots,A_r$ of sizes $a_1,\dots,a_r$, according to which tight component edges containing a given vertex and x are in. Note that $a_i \geq \delta_2(\mathcal{H}) \geq (1-\varepsilon)n/r$ for each $i \in [r]$. If these are the only tight components then, since each vertex is met by at least $r-1$ of them, some component meets at least $n(r-1)/r$ vertices. So we may assume that there is another tight component $\mathcal{B}$ which does not meet x. Suppose $\mathcal{B}$ meets b_i vertices in A_i for each i, and in total meets b vertices. Write $S = \{i \in [r] : b_i > 0\}$.

If $|S| \leq 2$ then the codegree of any pair which is in an edge of $\mathcal{B}$ and meets all the (at most 2) parts, is at most $\sum_{i\in S}(a_i - \delta_2(\mathcal{H})) \leq \sum_{i\in[r]}(a_i - \delta_2(\mathcal{H})) \leq n\varepsilon < n/(r+1) < \delta_2(\mathcal{H})$, giving a contradiction. So we may assume $|S| \geq 3$.

Claim 2. For each pair $i, j \in S$,

$$3\delta_2(\mathcal{H}) \leq b - b_i - b_j + a_i + a_j\,. \tag{5}$$

Proof of Claim 2. First, suppose that there are vertices $v_i \in A_i$ and $v_j \in A_j$ such that edges containing v_i, v_j are in $\mathcal{B}$. Then the codegree of this pair is at most $\sum_{S\ni k\neq i,j} b_k + (a_i - \delta_2(\mathcal{H})) + (a_j - \delta_2(\mathcal{H}))$, since only the b_k vertices in A_k which meet $\mathcal{B}$ can form an edge with v_i, v_j if $k \neq i, j$, and only the vertices $w \in A_i$ for which there is no edge of the form xwv_i are available (and similarly for A_j). Since the codegree of x, v_i is at least $\delta_2(\mathcal{H})$, and all vertices which complete an edge with x, v_i lie in A_i, at most $a_i - \delta_2(\mathcal{H})$ vertices are available to form edges with v_i, v_j. Rearranging, and noting that $\sum_{k\in S} b_k = b$, gives the desired inequality.

Secondly, suppose that no such vertices exist. For every $k \neq i, j$ such that there exist vertices $v_i \in A_i$ and $v_k \in A_k$ which can be extended to an edge of $\mathcal{B}$ (write S_i for the set of such k), similar reasoning gives $3\delta_2(\mathcal{H}) \leq b - b_i - b_j - b_k + a_i + a_k$, since no vertex from A_j can extend the pair v_i, v_k to an edge. If none of these gives the desired inequality, we must have $a_k - a_j > b_k$ for each $k \in S_i$. For any $l \in S \setminus \{i, j\}$, picking v_l in $\mathcal{B}$ and A_l, the pair v_i, v_l must have codegree at most

$$\begin{aligned}
\sum_{k\in S_i\setminus\{l\}} b_k + (a_i - \delta_2(\mathcal{H})) + (a_l - \delta_2(\mathcal{H})) &< \sum_{k\in S_i\setminus\{l\}} (a_k - a_j) + (a_i - \delta_2(\mathcal{H})) + (a_k - \delta_2(\mathcal{H}))\\
&\leq \sum_{k\in S_i\cup\{i\}} \left(a_k - (1-\varepsilon)\frac{n}{r}\right)\\
&\leq \sum_{k\in[r]} \left(a_k - (1-\varepsilon)\frac{n}{r}\right)\\
&= n\varepsilon < n/(r+1) < \delta_2(\mathcal{H})\,,
\end{aligned}$$

a contradiction. This completes the proof of Claim 2. ■

Now, averaging inequality (5) over all pairs $i \neq j \in S$ we get

$$3\delta_2(\mathcal{H}) \leq b - \frac{2(|S|-1)b}{|S|(|S|-1)} + \frac{2(n-(r-|S|)\delta_2(\mathcal{H}))}{|S|} = \frac{r-s-2}{r-s}b + \frac{2n-2s\delta_2(\mathcal{H})}{r-s}\,,$$

where $s = r - |S|$. Rearranging this (noting that $r-s-2>0$) gives

$$\begin{aligned}
b &\geq \frac{(3r-s)\delta_2(\mathcal{H}) - 2n}{r-s-2}\\
&\geq \frac{(3r-s)(1-\varepsilon)n - 2nr}{r(r-s-2)}\,.
\end{aligned}$$

It suffices to show that

$$\frac{(3r-s)(1-\varepsilon)n - 2nr}{r(r-s-2)} \geq \frac{(1-3\varepsilon)n}{r-2}\,,$$

or equivalently

$$(r-2)\big((3r-s)(1-\varepsilon) - 2r\big) \geq r(r-s-2)(1-3\varepsilon)\,.$$

But

$$(r-2)\big((3r-s)(1-\varepsilon) - 2r\big) - r(r-s-2)(1-3\varepsilon) = 2s(1-(r+1)\varepsilon) > 0\,,$$

as required. □

We stated Theorem 7 for the range of ε for which the proof works. However, in this range we also have $\delta_2(\mathcal{H}) \geq (1-\varepsilon)\frac{n}{r} > \frac{n}{r+1}$ and so, by Theorem 7 for $r+1$, $\mathrm{tc}(\mathcal{H}) \geq \frac{n}{r-1}$. This gives a better bound than (1) for $\varepsilon > \frac{1}{3r-3}$. Consequently we have the following lower bounds for $f_3(x)$:

$$f_3(x) \geq \begin{cases} 1 & \text{if } x > \frac{1}{3}\,; \\ \frac{2}{3} & \text{if } \frac{8}{27} \leq x \leq \frac{1}{3}\,; \\ 9x-2 & \text{if } \frac{5}{18} \leq x \leq \frac{8}{27}\,; \\ \frac{1}{r-1} & \text{if } \frac{1}{r+1} \leq x \leq \left(\frac{3r-4}{3r-3}\right)\frac{1}{r} \text{ where } r \geq 3\,; \\ \frac{3rx-2}{r-2} & \text{if } \left(\frac{3r-4}{3r-3}\right)\frac{1}{r} \leq x \leq \frac{1}{r} \text{ where } r \geq 4\,. \end{cases}$$

Conversely, Theorem 4 gives the following upper bounds. Let $(r_i)_{i\geq 0}$ be the sequence of integers such that $r_i - 2$ is a prime power or 0, i.e. the sequence that begins $2, 3, 4, 5, 6, 7, 9, 10, 11, 13, 15, \ldots$, and let $q_i = \frac{r_i - 3 + \frac{2}{r_i - 1}}{r_i^2 - 3r_i + 3}$ Then for every $i \geq 0$ we have

$$f_3(x) \leq \frac{r_i - 1}{r_i^2 - 3r_i + 3} \quad \text{for } x \in (q_{i+1}, q_i]$$

(the case $i = 0$ being the trivial upper bound $f_3(x) \leq 1$).

Figure 1 shows our upper and lower bounds. The bounds coincide for $x \in \left\{\frac{5}{21}\right\} \cup \left[\frac{8}{27}, 1\right]$.

References

[1] P. Allen, J. Böttcher, O. Cooley, R. Mycroft, Tight cycles and regular slices in dense hypergraphs. *J. Combin. Theory Ser. A* **149** (2017), 30–100.

[2] R. H. Bruck and H. J. Ryser, The nonexistence of certain finite projective planes, *Canadian J. Math.* **1** (1949), 88–93.

[3] O. Cooley, M. Kang and C. Koch, The size of the giant component in random hypergraphs, *Random Structures & Algorithms*, to appear.

[4] O. Cooley, M. Kang and Y. Person, Largest components in random hypergraphs, *Combin. Probab. Comput.*, to appear.

[5] Z. Füredi, Maximum degree and fractional matchings in uniform hypergraphs, *Combinatorica* **1** (1981), 155–162.

[6] A. Georgakopoulos, J. Haslegrave, R. Montgomery and B. Narayanan, Hamiltonian surfaces in 3-graphs, in preparation.

[7] E. Győri, G. Y. Katona and N. Lemons, Hypergraph extensions of the Erdős-Gallai Theorem, *European J. Combin.* **58** (2016), 238–246.

[8] M. Kahle and B. Pittel, Inside the critical window for cohomology of random k-complexes, *Random Structures & Algorithms* **48** (2016), 102–124.

[9] G. Y. Katona and H. A. Kierstead, Hamiltonian chains in hypergraphs, *J. Graph Theory* **30** (1999), 205–212.

[10] C. W. H. Lam, L. Thiel and S. Swiercz, The non-existence of finite projective planes of order 10, *Canadian J. Math.* **41** (1989), 1117–1123.

[11] N. Linial and R. Meshulam, Homological connectivity of random 2-complexes, *Combinatorica* **26** (2006), 475–487.

[12] V. Rödl, A. Ruciński and E. Szemerédi, A Dirac-type theorem for 3-uniform hypergraphs. *Combin. Probab. Comput.* **15** (2006), 229–251.