MINIMALIST DESIGNS

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ABSTRACT. The iterative absorption method has recently led to major progress in the area of (hyper-)graph decompositions. Amongst other results, a new proof of the Existence conjecture for combinatorial designs, and some generalizations, was obtained. Here, we illustrate the method by investigating triangle decompositions: we give a simple proof that a triangle-divisible graph of large minimum degree has a triangle decomposition and prove a similar result for quasi-random host graphs.

1. INTRODUCTION

1.1. **Steiner triple systems and the decomposition threshold.** A famous theorem of Kirkman [19] from 1847 states that a Steiner triple system of order n exists if and only if $n \equiv 1, 3 \mod 6$. Here, a *Steiner triple system of order n* is a collection of 3-subsets of $[n]$ such that every 2-subset of $[n]$ is contained in exactly one of the 3-sets. More generally, given graphs G and F, we say that G has an *F-decomposition* if its edge set can be partitioned into copies of F. Clearly, a Steiner triple system of order n is equivalent to a K_3-decomposition of K_n. Observe that if a graph G admits a K_3-decomposition, then the number of edges of G must be divisible by 3, and all the vertex degrees of G must be even. We say that G is K_3-*divisible* if it has these properties. Clearly, not every K_3-divisible graph has a K_3-decomposition (e.g. C_6). In fact, to decide whether a given graph has a K_3-decomposition is NP-hard [6]. However, the following beautiful conjecture of Nash-Williams suggests that if the minimum degree of G is sufficiently large, then divisibility is not only necessary but also sufficient for the existence of a K_3-decomposition.

Conjecture 1.1 (Nash-Williams [25]). *For sufficiently large n, every K_3-divisible graph G on n vertices with $\delta(G) \geq 3n/4$ has a K_3-decomposition.*

The following class of extremal examples shows that the bound on the minimum degree would be best possible. Given any $k \in \mathbb{N}$, let G_1 and G_2 be vertex-disjoint $(6k+2)$-regular graphs with $|G_1| = |G_2| = 12k + 6$ and let G_3 be the complete bipartite graph between $V(G_1)$ and $V(G_2)$. Let $G := G_1 \cup G_2 \cup G_3$. (In the standard construction, each of G_1 and G_2 is a union of two disjoint cliques of size $6k + 3$.) Clearly, $\delta(G) = 3|G|/4 - 1$ and G is K_3-divisible. However, every triangle in G contains at least one edge from $G_1 \cup G_2$. Since $2e(G_1 \cup G_2) < e(G_3)$, G cannot have a K_3-decomposition.

For $n \in \mathbb{N}$, define $\delta(n)$ as the minimum of all natural numbers $d \in \mathbb{N}$ such that every K_3-divisible graph G on n vertices with $\delta(G) \geq d$ has a K_3-decomposition. Hence, Conjecture 1.1 is equivalent to saying that $\delta(n) \leq 3n/4$ for all sufficiently large n.

The *decomposition threshold of K_3* is defined as $\delta_{K_3} := \limsup_{n \to \infty} \frac{\delta(n)}{n}$. Conjecture 1.1 would imply that $\delta_{K_3} \leq 3/4$ (and equality would hold by the above example). More generally, the *F-decomposition threshold δ_F* can be defined analogously for any (hyper-)graph F we wish to decompose into. It is conjectured in [9] that for every graph F, we have $\delta_F \leq 1 - 1/(\chi(F) + 1)$ (but equality does not hold for every F). This is reminiscent of related results in Extremal Combinatorics. For example, the Erdős–Stone–Simonovits theorem says that any graph G with

$e(G) \geq (1 - 1/(\chi(F) - 1) + o(1))\binom{n}{2}$ contains F as a subgraph, and a theorem due to Alon and Yuster states that if $\delta(G) \geq (1 - 1/\chi(F) + o(1))n$ and $|F|$ divides $|G|$ then G has an F-factor (again, the latter bound is not optimal for every F, see [22]).

The main application of the iterative absorption method is to turn an approximate decomposition into an exact decomposition. To formalize this, we define the approximate decomposition threshold δ^{0+} to be the infimum of all $\delta \in [0, 1]$ with the following property: for all $\gamma > 0$, there exists $n_0 \in \mathbb{N}$ such that any graph G on $n \geq n_0$ vertices with $\delta(G) \geq \delta n$ contains a K_3-decomposable subgraph H such that $\Delta(G - H) \leq \gamma n$. In this paper, we prove the following theorem, which reduces the challenge of showing that $\delta_{K_3} = 3/4$ to showing that $\delta^{0+} \leq 3/4$. The result itself is already contained in [4]. However, the proof we present here is much simpler and serves as an illustration as to how to use iterative absorption for decomposition problems.

Theorem 1.2. *Let $\delta := \max\{3/4, \delta^{0+}\}$. For all $\varepsilon > 0$, there exists $n_0 \in \mathbb{N}$ such that every K_3-divisible graph G on $n \geq n_0$ vertices with $\delta(G) \geq (\delta + \varepsilon)n$ has a K_3-decomposition.*

In a nutshell, Theorem 1.2 says that $\delta_{K_3} \leq \max\{3/4, \delta^{0+}\}$. Any improvement on the value of δ^{0+} thus immediately improves the value of δ_{K_3}. It follows from results in [7, 12] that $\delta^{0+} \leq 0.9$. Hence, an immediate corollary of Theorem 1.2 is that $\delta_{K_3} \leq 0.9$. It would be very desirable to prove that $\delta^{0+} \leq 3/4$, which would thus give an asymptotic version of Conjecture 1.1. (We will discuss δ^{0+} further in Section 1.2.) In order to prove Conjecture 1.1 in full, however, would in addition likely require a stability analysis — a daunting prospect given the abundance of extremal examples. On the other hand, for decompositions into even cycles (except of length 6), this has been carried out in [29]. Further results in the spirit of Theorem 1.2 for decompositions into arbitrary graphs F were obtained in [4, 9]. In particular, in [9] δ_F is determined for every bipartite graph F, and it is shown that the threshold of cliques equals its fractional version.

1.2. **Fractional decompositions.** A successful approach to obtain bounds on δ^{0+} is to take a detour via fractional decompositions. Let $\binom{G}{F}$ be the set of copies of F in G. A *fractional F-decomposition of G* is a function $\omega \colon \binom{G}{F} \to [0, 1]$ such that for all $e \in E(G)$,

$$(1.1) \qquad \sum_{F' \in \binom{G}{F} \colon e \in E(F')} \omega(F') = 1.$$

Thus, an F-decomposition is a fractional F-decomposition with image $\{0, 1\}$.

We define the *fractional decomposition threshold δ^* of K_3* to be the infimum of all $\delta \in [0, 1]$ with the following property: there exists $n_0 \in \mathbb{N}$ such that any graph G on $n \geq n_0$ vertices with $\delta(G) \geq \delta n$ has a fractional K_3-decomposition. Observe that the extremal example after Conjecture 1.1 also implies that $\delta^* \geq 3/4$.

Haxell and Rödl [12] showed that if G is an n-vertex graph with a fractional F-decomposition, then all but $o(n^2)$ edges of G can be covered with edge-disjoint copies of F. Note that the definition of δ^{0+} requires a leftover of small maximum degree, whereas this result only provides a leftover with $o(n^2)$ edges. It is however easy to turn such a leftover into one with small maximum degree (see e.g. Lemma 10.6 in [4]). Thus, $\delta^{0+} \leq \delta^*$.

The best bound on δ^* was obtained by Dross [7], who showed that $\delta^* \leq 0.9$, using an elegant approach based on the max-flow-min-cut theorem. The best current bound on the fractional decomposition threshold of larger cliques was proved in [24]. It would be very interesting to improve these results.

1.3. **Quasi-random host graphs.** Rather than graphs of large minimum degree, it also makes sense to consider quasi-random host graphs. A natural notion of quasi-randomness in this context is that of typicality. Given $p, \xi > 0$ and $h \in \mathbb{N}$, an n-vertex graph G is (ξ, h, p)-*typical* if for every set $A \subseteq V(G)$ with $|A| \leq h$ the common neighbourhood of the vertices in A has size $(1 \pm \xi)p^{|A|}n$. Note that a binomial random graph with edge probability p is likely to be typical. The following result (with h large) was first proved by Keevash [15].

Theorem 1.3. *For all $p > 0$, there exist $n_0 \in \mathbb{N}$ and $\xi > 0$ such that every $(\xi, 4, p)$-typical K_3-divisible graph on $n \geq n_0$ vertices has a K_3-decomposition.*

In Section 4, we will outline how the proof of Theorem 1.2 can be adapted to give a proof of Theorem 1.3.

1.4. **Designs.** More generally, it is natural to consider these question for uniform hypergraphs. In particular, such decompositions give rise to combinatorial designs with arbitrary parameters. More precisely, let F, H be r-graphs (i.e. r-uniform hypergraphs) and let K_n^r denote the complete r-graph on n vertices. An F-*decomposition* of H is a collection of copies of F in H such that every edge of H is contained in exactly one of these copies. A (q, r, λ)-*design* of H is a collection of distinct copies of K_q^r in H such that every edge of H is contained in exactly λ of these copies. (When $\lambda = 1$ and $H = K_n^r$, these are referred to as *Steiner systems*.)

The existence of (q, r, λ)-designs of K_n^r for fixed q, r, λ was proved by Keevash [15] (subject to n satisfying the necessary divisibility conditions). Much more generally, he proved the existence of (q, r, λ)-designs of H, where H is quasi-random (and dense). His proof is based on algebraic and probabilistic techniques (see [16] for an exposition of the triangle case and [18] for further discussions).

A new proof (based on iterative absorption) of this result was given in [10]. Moreover, [10] also provides bounds on the decomposition threshold of K_q^r. These methods and results were extended to F-designs for arbitrary r-graphs F in [11]. Further results (including the existence of resolvable designs and a new proof of the existence of F-designs) were subsequently proved by Keevash [17].

1.5. **Iterative absorption.** The main idea of the absorbing technique is quite natural: suppose we want to find some spanning structure in a graph or hypergraph, for instance a perfect matching, a Hamilton cycle, or an F-factor. In many such cases, it is much easier to find an 'almost-spanning' structure, e.g. a matching which covers almost all the vertices. The idea of the absorbing technique is that before finding the almost-spanning structure we set aside an absorbing structure which is capable of 'absorbing' the leftover vertices into the almost-spanning structure to obtain the desired spanning structure. Such an approach was introduced systematically in the influential paper by Rödl, Ruciński and Szemerédi [28] to prove an analogue of Dirac's theorem for 3-graphs (but goes back further than this, see e.g. the work of Krivelevich [21] on triangle factors in random graphs, and the result of Erdős, Gyárfás and Pyber [8] on vertex coverings with monochromatic cycles). Since then, the absorbing technique has been successfully applied to a wealth of problems concerning spanning structures. Of course, the success of the approach stands and falls with the ability to find this 'magic' absorbing structure. One key factor in this is the number of possible leftover configurations. Intuitively, the more possible leftover configurations there are, the more difficult it is to find an absorbing structure which can deal with all of them. Loosely speaking, this makes it much harder (if not impossible) to directly apply an absorbing technique for edge-decomposition problems (see e.g. [4, p. 343] for a back-of-the-envelope calculation).

The 'iterative absorption' method overcomes this issue by splitting up the absorbing process into many steps. In each step, the number of possible leftover configurations is drastically reduced using a 'partial absorbing procedure', until eventually one has sufficient control over the leftover to absorb it completely in a final absorption step. This approach was first used in [23] to find Hamilton decompositions of regular robust expanders. An iterative procedure using partial absorbers was also used in [20] to find optimal Hamilton packings in random graphs (though strictly speaking this is not a decomposition result). In the context of F-decompositions, the method was first applied in [4] to find F-decompositions of graphs of suitably high minimum degree. The results from [4] were strengthened in [9]. Even though the overall proof in [9] is more involved, the iterative absorption procedure itself is simpler than in [4]. The method has also been successfully applied to verify the Gyárfás-Lehel tree packing conjecture for bounded degree trees [14], as well as to find decompositions of dense graphs in the partite setting [5]. Last but not least, as mentioned in Section 1.4, the method was developed for hypergraphs (and thus designs) in [10, 11].

2. Preliminaries

For a graph G, we let $|G|$ denote the number of vertices of G and $e(G)$ the number of edges of G. We will sometimes identify a graph with its edge set if this enhances readability and does not affect the argument. For a vertex $v \in V(G)$, we write $N_G(v)$ for the neighbourhood of v and $d_G(v)$ for its degree. More generally, for a subset $X \subseteq V(G)$, we let $d_G(x, X)$ denote the number of neighbours of x in X. Let G be a graph and let X, Y be disjoint subsets of $V(G)$. We write $G[X]$ for the subgraph of G induced by X, and $G[X, Y]$ for the bipartite subgraph of G induced by X, Y. If G is a graph and H is a subgraph of G, then $G - H$ denotes the graph with vertex set $V(G)$ and edge set $E(G) \setminus E(H)$.

We write $[n]$ for the set $\{1, \ldots, n\}$. The expression $a = b \pm c$ means that $a \in [b - c, b + c]$. We write $x \ll y$ to mean that for any $y \in (0, 1]$ there exists an $x_0 \in (0, 1)$ such that for all $x \leq x_0$ the subsequent statement holds. Hierarchies with more constants are defined in a similar way and are to be read from the right to the left. We will always assume that the constants in our hierarchies are reals in $(0, 1]$. Moreover, if $1/x$ appears in a hierarchy, this implicitly means that x is a natural number. More precisely, $1/x \ll y$ means that for any $y \in (0, 1]$ there exists an $x_0 \in \mathbb{N}$ such that for all $x \in \mathbb{N}$ with $x \geq x_0$ the subsequent statement holds.

Let $m, n, N \in \mathbb{N}$ with $\max\{m, n\} < N$. Recall that a random variable X has hypergeometric distribution with parameters N, n, m if $X := |S \cap [m]|$, where S is a random subset of $[N]$ of size n. We write $X \sim Bin(n, p)$ if X has binomial distribution with parameters n, p. We will often use the following Chernoff-type bound.

Lemma 2.1 (see [13, Corollary 2.3, Remark 2.5, Theorem 2.8 and Theorem 2.10]). *Let X be the sum of n independent Bernoulli random variables or let X have a hypergeometric distribution with parameters N, n, m. Then the following hold.*

  (i) *For all $t \geq 0$, $\mathbb{P}[|X - \mathbb{E}[X]| \geq t] \leq 2\mathrm{e}^{-2t^2/n}$.*
  (ii) *For all $0 \leq \varepsilon \leq 3/2$, $\mathbb{P}[X \neq (1 \pm \varepsilon)\mathbb{E}[X]] \leq 2\mathrm{e}^{-\varepsilon^2\mathbb{E}[X]/3}$.*

3. Proof of Theorem 1.2

3.1. **The final absorbers.** In this subsection, we construct the absorbers which will be set aside initially and then used in the final absorption step. As discussed below, our absorbers will consist of the union of 'exclusive' absorbers which can absorb a given graph L. Here L is a leftover from a previous partial decomposition step.

Definition 3.1 (Absorber for L). Given a graph L, an *absorber for L* is a graph A such that $V(L) \subseteq V(A)$ is independent in A and both A and $A \cup L$ have a K_3-decomposition.

Note that the condition that $V(L)$ is independent in A implies that A and L are edge-disjoint. Observe that if A is an absorber for L, then in particular, both A and $A \cup L$ are K_3-divisible, and thus L must be K_3-divisible. Conversely, we will show that for any K_3-divisible graph L, there exists an absorber.

To guarantee that we can actually find the constructed absorbers in a given host graph G of large degree, we will construct absorbers which have low degeneracy. For a graph H and a subset $U \subseteq V(H)$, the *degeneracy of H rooted at U* is the smallest $d \in \mathbb{N} \cup \{0\}$ such that there exists an ordering $v_1, \ldots, v_{|H|-|U|}$ of the vertices of $V(H) \setminus U$ such that for all $i \in [|H| - |U|]$,

$$d_H(v_i, U \cup \{v_j \, : \, 1 \leq j < i\}) \leq d.$$

The goal of this subsection is to prove the following lemma.

Lemma 3.2. *Let L be any K_3-divisible graph. There exists an absorber A for L such that the degeneracy of A rooted at $V(L)$ is at most 4.*

We construct absorbers as a concatenation of 'transformers', whose purpose is, roughly speaking, to transform a given leftover from a 'partial' K_3-decomposition into a new leftover. The goal of course is to eventually transform the given leftover L into a new leftover which is K_3-decomposable.
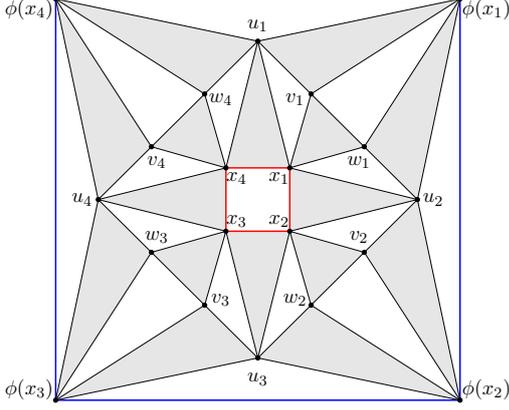
FIGURE 1. A (C_4, C_4)-transformer.
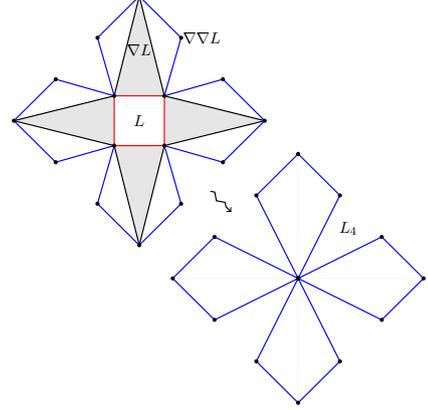
FIGURE 2. The canonical graph L_4.

Definition 3.3 (Transformer). Given vertex-disjoint graphs L, L', an (L, L')-transformer is a graph T such that $V(L \cup L') \subseteq V(T)$ is independent in T and both $T \cup L$ and $T \cup L'$ have a K_3-decomposition.

Note that the condition that $V(L \cup L')$ is independent in T implies that T is edge-disjoint from both L and L'. Observe also that if T is an (L, L')-transformer and L' is K_3-decomposable, then $A := T \cup L'$ is an absorber for L.

Given graphs H and H', a function $\phi \colon V(H) \to V(H')$ is an *edge-bijective homomorphism from H to H'* if $\phi(x)\phi(y) \in E(H')$ for all $xy \in E(H)$, and $e(H) = e(H') = |\{\phi(x)\phi(y) \ : \ xy \in E(H)\}|$. We write $H \rightsquigarrow H'$ if such a function ϕ exists. More loosely, if $H \rightsquigarrow H'$ then we can merge vertices together in H without creating multi-edges to get a copy of H'.

Lemma 3.4. *Let L and L' be vertex-disjoint graphs such that $L \rightsquigarrow L'$ and $2 \mid d_L(x)$ for all $x \in V(L)$. There exists an (L, L')-transformer T such that the degeneracy of T rooted at $V(L \cup L')$ is at most 4.*

Proof. Let $\phi \colon L \to L'$ be an edge-bijective homomorphism. Since $2 \mid d_L(x)$ for all $x \in V(L)$, there exists a decomposition $\mathcal{C}$ of L into cycles. Note that L' decomposes into the graphs $\{\phi(C)\}_{C \in \mathcal{C}}$. Suppose that we can find a $(C, \phi(C))$-transformer T_C for every $C \in \mathcal{C}$ such that the degeneracy of T_C rooted at $V(C \cup \phi(C))$ is at most 4. We may clearly assume that $V(T_C) \cap V(L \cup L') = V(C \cup \phi(C))$ for all $C \in \mathcal{C}$ and $V(T_C) \cap V(T_{C'}) \subseteq V(L \cup L')$ for all $C, C' \in \mathcal{C}$. It is then easy to see that $T := \bigcup_{C \in \mathcal{C}} T_C$ is an (L, L')-transformer such that the degeneracy of T rooted at $V(L \cup L')$ is at most 4.

For the remainder of the proof, we may thus assume that L is a cycle $x_1 x_2 \ldots x_s x_1$. We can then construct T as follows. Let $\{u_i, v_i, w_i \ : \ i \in [s]\}$ be a set of $3s$ vertices disjoint from $V(L \cup L')$. The vertex set of T will be $V(L \cup L') \cup \{u_i, v_i, w_i \ : \ i \in [s]\}$. Moreover, we define the following sets of edges (indices modulo s):

$$E := \{x_i u_i, x_i v_i, x_i w_i, x_i u_{i+1} \ : \ i \in [s]\};$$
$$E' := \{\phi(x_i)u_i, \phi(x_i)v_i, \phi(x_i)w_i, \phi(x_i)u_{i+1} \ : \ i \in [s]\};$$
$$\tilde{E} := \{u_i v_i, w_i u_{i+1} \ : \ i \in [s]\};$$
$$E^* := \{v_i w_i \ : \ i \in [s]\}.$$

Let $E(T) := E \cup E' \cup \tilde{E} \cup E^*$ (cf. Figure 1).

We claim that T is the desired transformer. Clearly, $V(L \cup L') \subseteq V(T)$ is independent in T. Moreover, to see that the degeneracy of T rooted at $V(L \cup L')$ is at most 4, order the vertices of $\{u_i, v_i, w_i \ : \ i \in [s]\}$ such that $u_1, \ldots, u_s$ come first. Finally, note that all of $E(L) \cup E \cup E^*$, $E(L') \cup E' \cup E^*$, $E \cup \tilde{E}$, $E' \cup \tilde{E}$ have a K_3-decomposition. This implies that both $T \cup L$ and $T \cup L'$ have a K_3-decomposition. $\qquad \square$

We remark that there is a slightly simpler construction of an (L, L')-transformer T if we allowed the degeneracy to be 6 (with the effect of obtaining 5/6 instead of the desired 3/4 in Theorem 1.2). Simply add for every edge $e \in E(L)$ a vertex z_e and join z_e to the endvertices of e and $\phi(e)$. Then, for every vertex $x \in V(L)$, add a perfect matching on the vertices $\{z_{xy} : y \in N_L(x)\}$. This construction turns out to be more easily generalisable to obtain transformers when considering F-decompositions rather than K_3-decompositions.

As mentioned before, transformers enable us to transform a given leftover from a 'partial' K_3-decomposition into a new leftover. We now define a 'canonical' graph L_m which any m-edge K_3-divisible graph L can be transformed into (via an intermediary denoted $\nabla\nabla L$). Applying this to both an arbitrary such graph L and a clearly K_3-decomposable m-edge graph L', and combining the resulting transformers through L_m, will give an absorber for L.

For $m \in \mathbb{N}$, let L_m be the graph with

$$V(L_m) = \{v^*, v_1, \ldots, v_{3m}\}, \quad E(L_m) = \bigcup_{i \in [m]} \{v^* v_{3i-2}, v_{3i-2} v_{3i-1}, v_{3i-1} v_{3i}, v_{3i} v^*\}.$$

That is, L_m is obtained from C_{4m} by taking every fourth vertex on the cycle and identifying all these vertices to a single vertex v^*. Note that if L is any graph with m edges, then replacing every edge with a path of length 4 and identifying the original vertices of L yields a graph isomorphic to L_m (cf. Fact 3.5 and Figure 2). In this sense, L_m will serve as a 'canonical' graph. Instead of L_m, we could have also worked with C_{4m} directly, but this would not generalize to F-decompositions for other graphs F, while the current approach does generalize in a natural way.

More formally, given a graph L, we define $\tilde\nabla L$ as the graph obtained from L by adding a new set $\{z_e : e \in E(L)\}$ of vertices disjoint from $V(L)$, and for every edge $e \in E(L)$, we join both endpoints of e to z_e. In other words, we extend every edge of L into a copy of K_3. Obviously, $\tilde\nabla L$ has a K_3-decomposition. Furthermore, let $\nabla L := \tilde\nabla L - L$. Note that ∇L is obtained from L by replacing every edge with a path of length 2, and $\nabla\nabla L$ is obtained from L by replacing every edge with a path of length 4.

Fact 3.5. *For every graph L with $m := e(L)$, we have $\nabla\nabla L \rightsquigarrow L_m$.*

Proof. Define $\phi \colon \nabla\nabla L \to L_m$ such that $\phi(x) := v^*$ for all $x \in V(L)$, and map $V(\nabla\nabla L) \setminus V(L)$ bijectively to $V(L_m) \setminus \{v^*\}$ in the obvious way. $\square$

We now construct the absorber A as the union of several intermediate graphs and transformers.

Proof of Lemma 3.2. Let $m := e(L)$. Let L' be the vertex-disjoint union of $m/3$ triangles. Moreover, let L_m be the 'canonical' graph defined above, and also define $\nabla L, \nabla\nabla L, \nabla L', \nabla\nabla L'$ as above. We assume that $\nabla\nabla L$, $\nabla\nabla L'$ and L_m are pairwise vertex-disjoint.

By Fact 3.5, we have $\nabla\nabla L \rightsquigarrow L_m$ and $\nabla\nabla L' \rightsquigarrow L_m$. Thus, by Lemma 3.4, there exists an $(\nabla\nabla L, L_m)$-transformer T such that the degeneracy of T rooted at $V(\nabla\nabla L \cup L_m)$ is at most 4, and there exists an $(\nabla\nabla L', L_m)$-transformer T' such that the degeneracy of T' rooted at $V(\nabla\nabla L' \cup L_m)$ is at most 4. We may assume that T and T' consist of new vertices except for the unavoidable intersections, that is, $V(T) \cap V(T') = V(L_m)$, $V(T) \cap V(\nabla\nabla L') = \emptyset$, $V(T') \cap V(\nabla\nabla L) = \emptyset$.

We now define the graph

$$A := \nabla L \cup \nabla\nabla L \cup T \cup L_m \cup T' \cup \nabla\nabla L' \cup \nabla L' \cup L'$$

and claim that A is the desired absorber. Clearly, $V(L)$ is independent in A, and it is easy to see that the degeneracy of A rooted at $V(L)$ is at most 4.

Finally, note that $\nabla L \cup \nabla\nabla L = \tilde\nabla(\nabla L)$, $T \cup L_m$, $T' \cup \nabla\nabla L'$ and $\nabla L' \cup L' = \tilde\nabla L'$ are pairwise edge-disjoint and are all K_3-decomposable. Thus, A has a K_3-decomposition. Similarly,

$$A \cup L = (L \cup \nabla L) \cup (\nabla\nabla L \cup T) \cup (L_m \cup T') \cup (\nabla\nabla L' \cup \nabla L') \cup L'$$

has a K_3-decomposition. $\square$

3.2. **Vortices.** Our proof method involves an iterative absorption process, where in each iteration, the number of leftover configurations is drastically reduced. We ensure this by forcing leftover edges into smaller and smaller subsets of the vertex set. The underlying structure is a 'vortex', which is defined as follows.

Definition 3.6 (Vortex). Let G be a graph on n vertices. A (δ, ε, m)-*vortex in G* is a sequence $U_0 \supseteq U_1 \supseteq \cdots \supseteq U_\ell$ such that

(V1) $U_0 = V(G)$;
(V2) $|U_i| = \lfloor \varepsilon |U_{i-1}| \rfloor$ for all $i \in [\ell]$;
(V3) $|U_\ell| = m$;
(V4) $d_G(x, U_i) \geq \delta |U_i|$ for all $i \in [\ell]$ and $x \in U_{i-1}$.

We now show that every large graph of high minimum degree contains a vortex such that the final set U_ℓ has constant size. This can easily be achieved by successively taking random subsets.

Lemma 3.7. *Let $\delta \in [0,1]$ and $1/m' \ll \varepsilon < 1$. Suppose that G is a graph on $n \geq m'$ vertices with $\delta(G) \geq \delta n$. Then G has a $(\delta - \varepsilon, \varepsilon, m)$-vortex for some $\lfloor \varepsilon m' \rfloor \leq m \leq m'$.*

Proof. Recursively, define $n_0 := n$ and $n_i := \lfloor \varepsilon n_{i-1} \rfloor$. Observe that $\varepsilon^i n \geq n_i \geq \varepsilon^i n - 1/(1-\varepsilon)$. Let $\ell := 1 + \max\{i \geq 0 : n_i \geq m'\}$ and let $m := n_\ell$. Note that $\lfloor \varepsilon m' \rfloor \leq m \leq m'$. Further, for $i \in [\ell]$, let

$$(3.1) \qquad \varepsilon_i := n^{-1/3} \sum_{j=1}^{i} \varepsilon^{-(j-1)/3} = n^{-1/3} \frac{\varepsilon^{-i/3} - 1}{\varepsilon^{-1/3} - 1} \leq \frac{(\varepsilon^{i-1} n)^{-1/3}}{1 - \varepsilon^{1/3}} \leq \frac{m'^{-1/3}}{1 - \varepsilon^{1/3}} \leq \frac{\varepsilon}{3},$$

where we have used $\varepsilon^{i-1} n \geq \varepsilon^{\ell-1} n \geq n_{\ell-1} \geq m'$, and let $\varepsilon_0 := 0$.

Now, suppose that for some $i \in [\ell]$, we have already found a $(\delta - 3\varepsilon_{i-1}, \varepsilon, n_{i-1})$-vortex $U_0, \ldots, U_{i-1}$ in G (which is true for $i = 1$). In particular, $\delta(G[U_{i-1}]) \geq (\delta - 3\varepsilon_{i-1}) n_{i-1}$. Choose U_i to be a random subset of U_{i-1} of size n_i. Lemma 2.1 implies that with positive probability, $d_G(x, U_i) \geq (\delta - 3\varepsilon_{i-1} - 2n_{i-1}^{-1/3}) n_i$ for all $x \in U_{i-1}$. Fix such a choice of U_i. Then $U_0, \ldots, U_i$ form a $(\delta - 3\varepsilon_i, \varepsilon, n_i)$-vortex in G. Repeating this for all $i \in [\ell]$, we finally obtain a $(\delta - 3\varepsilon_\ell, \varepsilon, m)$-vortex $U_0, \ldots, U_\ell$ in G. As, by (3.1), $\varepsilon_\ell \leq \varepsilon/3$, the lemma follows. $\square$

3.3. **Cover down lemma.** The engine behind the iterative absorption procedure is the following 'Cover down lemma'. Recall that the definition of δ^{0+} allows us to find approximate K_3-decompositions such that the maximum degree of the leftover is very small. The strength of the Cover down lemma is that it also forces all leftover edges to lie inside a prescribed small vertex set U (which one might think of as the next vortex set U_i in the sequence).

Lemma 3.8 (Cover down lemma). *Suppose $1/n \ll \varepsilon$ and let $\delta := \max\{2/3, \delta^{0+}\}$. Let G be a graph on n vertices and $U \subseteq V(G)$ with $|U| = \lfloor \varepsilon n \rfloor$. Suppose that $\delta(G) \geq (\delta + 3\varepsilon) n$ and $d_G(x, U) \geq (\delta + 2\varepsilon)|U|$ for all $x \in V(G)$. Also assume that $d_G(x)$ is even for all $x \in V(G) \setminus U$. Then there exists a K_3-decomposable subgraph $H \subseteq G$ such that $G - G[U] \subseteq H$ and $\Delta(H[U]) \leq \varepsilon^{10} n$.*

In order to prove Lemma 3.8, we will first randomly select a sparse reserve graph before using the definition of δ^{0+} to find an approximate K_3-decomposition of $G - G[U]$. We then cover all leftover edges which are not inside U in two stages. First, we find for every leftover edge $e = xy$ with $x, y \in V(G) \setminus U$ a vertex $u_e \in U$ such that u_e, x, y form a triangle in G. In a second step, we cover the remaining 'crossing' leftover edges. Suppose $x \in V(G) \setminus U$ and let U_x be the set of all remaining neighbours of x. What we would now like to find is a perfect matching M_x of the 'link graph' $G[U_x]$ of x. Clearly, the edges of M_x together with all the edges from U_x to x would then form edge-disjoint triangles covering all remaining edges at x. Lemma 3.10 will allow us to achieve the latter simultaneously for all $x \in V(G) \setminus U$. For this, we need the following simple result.

Fact 3.9 (cf. [26, Lemma 8])**.** *Let $X_1, \ldots, X_n$ be Bernoulli random variables such that for all $i \in [n]$, we have $\mathbb{P}\left[X_i = 1 \mid X_1, \ldots, X_{i-1}\right] \le p$. Let $B \sim Bin(n, p)$ and $X := \sum_{i=1}^{n} X_i$. Then $\mathbb{P}\left[X \ge a\right] \le \mathbb{P}\left[B \ge a\right]$ for all $a \ge 0$.*

Lemma 3.10. *Let $1/n \ll \rho$ and $N \in \mathbb{N}$. Let G be a graph on n vertices and suppose there are sets $U_1, \ldots, U_N \subseteq V(G)$ with the following properties:*

- *(i) $2 \mid |U_i|$ and $\delta(G[U_i]) \ge (1/2 + 4\rho^{1/6})|U_i|$ for all $i \in [N]$;*
- *(ii) $|U_i| \ge \rho^{4/3} n$ for all $i \in [N]$;*
- *(iii) $|U_i \cap U_j| \le \rho^2 n$ for all $1 \le i < j \le N$;*
- *(iv) every vertex $u \in V(G)$ is contained in at most ρn of the sets U_i.*

Then for every $i \in [N]$, there exists a perfect matching M_i of $G[U_i]$, such that all the matchings $\{M_i\}_{i \in [N]}$ are pairwise edge-disjoint.

It follows directly from (i) that for every $i \in [N]$, there exists a perfect matching M_i of $G[U_i]$. The difficulty here lies in finding edge-disjoint ones. For this, we use a randomised algorithm.

Proof. Let $t := \lceil 2\rho^{3/2} n \rceil$ and define $G_i := G[U_i]$ for all $i \in [N]$. Suppose that we have already found $M_1, \ldots, M_{i-1}$ for some $i \in [N]$. We now define M_i as follows. Let $H_{i-1} := \bigcup_{j=1}^{i-1} M_j$ and let $G'_i := (G - H_{i-1})[U_i]$. If $\Delta(H_{i-1}[U_i]) \le \rho^{3/2} n$, then

$$\delta(G'_i) \ge \delta(G[U_i]) - \Delta(H_{i-1}[U_i]) \ge (1/2 + 4\rho^{1/6})|U_i| - \rho^{3/2} n \ge |U_i|/2 + t$$

by (i) and (ii). Thus, we can successively find t edge-disjoint perfect matchings $A_1, \ldots, A_t$ of G'_i (which are all suitable candidates for M_i). Otherwise, if $\Delta(H_{i-1}[U_i]) > \rho^{3/2} n$, then let $A_1, \ldots, A_t$ be empty graphs on U_i.

In either case, we have found edge-disjoint subgraphs $A_1, \ldots, A_t$ of G'_i. Pick $s \in [t]$ uniformly at random and set $M_i := A_s$. The lemma follows if the following holds with positive probability:

$$(3.2) \qquad\qquad \Delta(H_{i-1}[U_i]) \le \rho^{3/2} n \quad \text{for all } i \in [N].$$

For $i \in [N]$ and $u \in U_i$, let $J^{i,u}$ be the set of indices $j \in [i-1]$ such that $u \in U_j$, and for $j \in J^{i,u}$, let $Y_j^{i,u}$ be the indicator variable of the event that $uu' \in E(M_j)$ for some $u' \in U_i$. Observe that

$$d_{H_{i-1}[U_i]}(u) = \sum_{j \in J^{i,u}} Y_j^{i,u}.$$

Now, fix $i \in [N]$ and $u \in U_i$. Crucially, for any $j \in J^{i,u}$, by (iii), at most $\rho^2 n$ of the subgraphs $A_1 \ldots, A_t$ that we picked in G'_j contain an edge incident to u in G_i (regardless of the previous choices). Let $j_1, \ldots, j_{|J^{i,u}|}$ be the enumeration of $J^{i,u}$ in increasing order. By the above, for all $\ell \in [|J^{i,u}|]$, we have

$$\mathbb{P}\left[Y_{j_\ell}^{i,u} = 1 \mid Y_{j_1}^{i,u}, \ldots, Y_{j_{\ell-1}}^{i,u}\right] \le \frac{\rho^2 n}{t} \le \frac{\rho^{1/2}}{2}.$$

Let $B \sim Bin(|J^{i,u}|, \rho^{1/2}/2)$. Since $|J^{i,u}| \le \rho n$ by (iv), we have $\mathbb{E}[B] \le \rho^{3/2} n / 2$. Using Fact 3.9 and Lemma 2.1, we infer that

$$\mathbb{P}\left[\sum_{j \in J^{i,u}} Y_j^{i,u} > \rho^{3/2} n\right] \le \mathbb{P}\left[B > \rho^{3/2} n\right] \le \mathbb{P}\left[B > \mathbb{E}[B] + \rho^{3/2} n / 2\right] \le 2e^{-\rho^2 n / 2}.$$

Finally, since by (iv) there are at most ρn^2 pairs (i, u) with $u \in U_i$, a union bound implies that (3.2) holds with positive probability. $\qquad\square$

Proof of Lemma 3.8. Choose new constants $\gamma, \rho > 0$ such that $1/n \ll \gamma \ll \rho \ll \varepsilon$. Let $W := V(G) \setminus U$ and let $w_1, \ldots, w_N$ be an enumeration of W.

We first observe that since $d_G(x, U) \geq (2/3 + 2\varepsilon)|U|$ for all $x \in V(G)$, it follows that for all $x, y \in V(G)$, we have

$$(3.3) \qquad |N_G(x) \cap N_G(y) \cap U| \geq (1/2 + 3\varepsilon)|N_G(y) \cap U| \geq |U|/3.$$

Before obtaining an approximate decomposition, we set aside a sparse graph R which will act as a 'partial absorber'. For this, let $U_1, \ldots, U_N$ be sets with the following properties:

(a) $U_i \subseteq N_G(w_i) \cap U$ for all $i \in [N]$;
(b) $|U_i| = (1 \pm \rho)\rho |N_G(w_i) \cap U|$ for all $i \in [N]$;
(c) $\rho^2 |U|/4 \leq |U_i \cap U_j| \leq 2\rho^2 |U|$ for all $1 \leq i < j \leq N$;
(d) $|N_G(u) \cap U_i| \geq (1 - \rho)\rho(1/2 + 3\varepsilon)|N_G(w_i) \cap U|$ for all $u \in U$ and $i \in [N]$;
(e) each $u \in U$ is contained in at most $2\rho n$ of the U_i's.

That such subsets exist can be seen via a probabilistic argument. Indeed, for every pair (u, i) with $i \in [N]$ and $u \in N_G(w_i) \cap U$, include u in U_i with probability ρ (independently of all other pairs). Applying Lemma 2.1 (and using (3.3)) shows that the random sets $U_1, \ldots, U_N$ satisfy the desired properties with positive probability. Note that it follows from (b) and (d) that for all $u \in U$ and $i \in [N]$, we have

$$(3.4) \qquad d_G(u, U_i) \geq (1/2 + 2\varepsilon)|U_i|.$$

Let R be the subgraph of $G[U, W]$ consisting of all edges uw_i where $i \in [N]$ and $u \in U_i$. Note that $\Delta(R) \leq 2\rho n$ by (b) and (e).

Let $G' := G - G[U] - R$. Clearly, we have $\delta(G') \geq (\delta^{0+} + \varepsilon)n$. By definition of δ^{0+}, there exists a subgraph L of G' such that $\Delta(L) \leq \gamma n$ and $G' - L$ is K_3-decomposable.

Next, for every edge $e = w_i w_j \in E(L[W])$, we choose a vertex $u_e \in U_i \cap U_j$ in such a way that $u_e \neq u_{e'}$ whenever $e \cap e' \neq \emptyset$.

This can be done greedily. Indeed, whenever we want to choose $u_{w_i w_j}$, there are at least $\rho^2 |U|/4$ vertices $u \in U_i \cap U_j$ by (c). Moreover, at most $2\Delta(L)$ of these vertices u are blocked by some edge e' which has w_i or w_j as an endpoint and $u_{e'} = u$ has been previously chosen. Since $2\Delta(L) < \rho^2 |U|/4$, we can always choose a suitable vertex $u_{w_i w_j}$.

Let $\hat{H}$ be the graph consisting of all the edges $u_{w_i w_j} w_i, u_{w_i w_j} w_j$ with $w_i w_j \in E(L[W])$. By (a), we can see that $\hat{H}$ is a subgraph of R. Moreover, $\hat{H} \cup L[W]$ clearly has a K_3-decomposition. Let

$$R' := (R - \hat{H}) \cup L[U, W].$$

By the above, $G - G[U] - R' = (G' - L) \cup (\hat{H} \cup L[W])$ has a K_3-decomposition. It remains to cover all the edges of R' using only a few edges of $G[U]$. For every $i \in [N]$, let

$$U_i' := N_{R'}(w_i) = (U_i \setminus N_{\hat{H}}(w_i)) \cup N_{L[U,W]}(w_i).$$

Since $G[U] \cup R'$ is obtained from G by removing edge-disjoint triangles, we have that $|U_i'|$ is even for all $i \in [N]$. Moreover, since $d_{\hat{H}}(w_i) = d_{L[W]}(w_i) \leq \gamma n$ and $d_{L[U,W]}(w_i) \leq \gamma n$ for all $i \in [N]$, we have $|U_i'| \geq \rho |U|/2$ for all $i \in [N]$ and $|U_i' \cap U_j'| \leq 3\rho^2 |U|$ for all $1 \leq i < j \leq N$. Since $d_{L[U,W]}(u) \leq \gamma n$ for all $u \in U$, it also follows that every vertex $u \in U$ is contained in at most $3\rho n \leq 4\rho \varepsilon^{-1}|U|$ of the U_i''s. Moreover, we can deduce from (3.4) that $\delta(G[U_i']) \geq (1/2 + \varepsilon)|U_i'|$ for all $i \in [N]$. Thus, by Lemma 3.10 (with $G[U], |U|, 4\rho/\varepsilon$ playing the roles of G, n, ρ), for every $i \in [N]$, there exists a perfect matching of $G[U_i']$, such that all the matchings $\{M_i\}_{i \in [N]}$ are pairwise edge-disjoint. Then

$$\bigcup_{i \in [N]} M_i \cup R' = \bigcup_{i \in [N]} (M_i \cup \{w_i u : u \in N_{R'}(w_i)\})$$

is K_3-decomposable. Thus $H := (G - G[U]) \cup \bigcup_{i \in [N]} M_i$ is K_3-decomposable. Moreover, $\Delta(H[U]) = \Delta(\bigcup_{i \in [N]} M_i) \leq 3\rho n \leq \varepsilon^{10} n$. $\qquad \square$

3.4. **Proof of Theorem 1.2.** We can now combine Lemmas 3.2, 3.7 and 3.8 to prove Theorem 1.2.

Proof of Theorem 1.2.   For convenience, we will prove this with 8ε in place of ε. That is, we assume that $\delta(G) \geq (\delta + 8\varepsilon)n$. Choose new constants $m', M \in \mathbb{N}$ such that $1/n_0 \ll 1/M \ll 1/m' \ll \varepsilon \ll 1$. Let $\delta := \max\{\delta^{0+}, 3/4\}$. By Lemma 3.2, and as $1/M \ll 1/m'$, for every K_3-divisible graph L with $|L| \leq m'$, there exists an absorber A_L for L such that $|A_L| \leq M$ and the degeneracy of A_L rooted at $V(L)$ is at most 4.

Let G be a K_3-divisible graph on $n \geq n_0$ vertices with $\delta(G) \geq (\delta + 8\varepsilon)n$. Our aim is to show that G has a K_3-decomposition. We achieve this in four steps.

Step 1. First, we apply Lemma 3.7 to obtain a $(\delta + 7\varepsilon, \varepsilon, m)$-vortex $U_0, U_1, \ldots, U_\ell$ in G for some $\lfloor \varepsilon m' \rfloor \leq m \leq m'$.

Step 2. Next, we find 'exclusive' absorbers for the possible leftover graphs on U_ℓ. To this end, let $\mathcal{L}$ be the collection of all spanning K_3-divisible subgraphs of $G[U_\ell]$. Obviously, $|\mathcal{L}| \leq 2^{\binom{m}{2}}$. It is thus easy to find edge-disjoint subgraphs $\{\tilde{A}_L\}_{L \in \mathcal{L}}$ of G such that for all $L \in \mathcal{L}$, we have that $\tilde{A}_L$ is an absorber for L, $|\tilde{A}_L| \leq M$, and $\tilde{A}_L[U_1]$ is empty. Indeed, we can find these graphs in turn. Suppose we want to find $\tilde{A}_L$. Consider the graph $\tilde{G}$ obtained from $G - G[U_1]$ by deleting the edges of previously chosen absorbers. Note that $\delta(\tilde{G}) \geq (3/4 + \varepsilon)n$. Thus, any four vertices in $\tilde{G}$ have at least $4\varepsilon n$ common neighbours. Since there is an ordering of the vertices of $V(A_L) \setminus V(L)$ such that every vertex is joined to at most 4 preceding vertices in A_L, we can embed the vertices of $V(A_L) \setminus V(L)$ one after the other into $\tilde{G}$ to obtain $\tilde{A}_L$.

Let $A^* := \bigcup_{L \in \mathcal{L}} \tilde{A}_L$. Observe that A^* has the following crucial property:

$$(3.5) \qquad \text{given any } K_3\text{-divisible subgraph } L^* \text{ of } G[U_\ell], A^* \cup L^* \text{ has a } K_3\text{-decomposition.}$$

Let $G' := G - A^*$. Observe that since $\Delta(A^*) \leq M|\mathcal{L}|$ and $A^*[U_1]$ is empty, we have that $U_0, U_1, \ldots, U_\ell$ is a $(\delta + 6\varepsilon, \varepsilon, m)$-vortex in G' and $\delta(G') \geq (\delta + 7\varepsilon)n$. Moreover, since A^* is the edge-disjoint union of absorbers, it must be K_3-divisible, and thus G' is also K_3-divisible.

Step 3. We now iteratively apply the Cover down lemma (Lemma 3.8) to cover all the edges of G' except possibly some inside U_ℓ. More precisely, we show inductively that for all $i \in [\ell] \cup \{0\}$, there exists a subgraph $G_i \subseteq G'[U_i]$ such that $G' - G_i$ has a K_3-decomposition, and such that the following hold (where $U_{\ell+1} := \emptyset$):

$$(3.6) \qquad\qquad\qquad \delta(G_i) \geq (\delta + 4\varepsilon)|U_i|;$$

$$(3.7) \qquad\qquad d_{G_i}(x, U_{i+1}) \geq (\delta + 5\varepsilon)|U_{i+1}| \text{ for all } x \in U_i;$$

$$(3.8) \qquad\qquad\qquad G_i[U_{i+1}] = G'[U_{i+1}].$$

Clearly, this holds for $i = 0$ with $G_0 := G'$. Now, suppose that for some $i \in [\ell - 1] \cup \{0\}$, we have found G_i satisfying the above. Note that G_i is K_3-divisible. Define $G'_i := G_i - G_i[U_{i+2}]$. We still have that $d_{G'_i}(x)$ is even for all $x \in U_i \setminus U_{i+1}$. Thus, by Lemma 3.8 (with G'_i, U_{i+1}, ε playing the roles of G, U, ε), there exists a K_3-decomposable subgraph $H \subseteq G'_i$ such that $G'_i - G'_i[U_{i+1}] \subseteq H$ and $\Delta(H[U_{i+1}]) \leq \varepsilon^8|U_{i+1}|$. Let $G_{i+1} := (G_i - H)[U_{i+1}]$.

Note that since $G_i - G_i[U_{i+1}] = G'_i - G'_i[U_{i+1}] \subseteq H$, we can deduce that $G' - G_{i+1} = (G' - G_i) \cup H$ is K_3-decomposable, as desired. Moreover, since $H[U_{i+2}]$ is empty by definition of G'_i, we clearly have $G_{i+1}[U_{i+2}] = G'[U_{i+2}]$. Observe that

$$\delta(G_{i+1}) \overset{(3.7)}{\geq} (\delta + 5\varepsilon)|U_{i+1}| - \Delta(H[U_{i+1}]) \geq (\delta + 4\varepsilon)|U_{i+1}|$$

and for every $x \in U_{i+1}$, we have

$$d_{G_{i+1}}(x, U_{i+2}) \overset{(3.8),(V4)}{\geq} (\delta + 6\varepsilon)|U_{i+2}| - \Delta(H[U_{i+1}]) \geq (\delta + 5\varepsilon)|U_{i+2}|.$$

Thus, (3.6)–(3.8) hold with i replaced by $i+1$. By induction, there exists a subgraph $G_\ell \subseteq G'[U_\ell]$ such that $G' - G_\ell$ has a K_3-decomposition.

Step 4. Finally, since G_ℓ is K_3-divisible, $A^* \cup G_\ell$ has a K_3-decomposition by (3.5). Altogether, $G = (G' - G_\ell) \cup (A^* \cup G_\ell)$ has a K_3-decomposition, as desired. $\square$

4. Proof of Theorem 1.3

The proof of Theorem 1.2 can be easily adapted to prove Theorem 1.3, modulo a 'boosting step' which we discuss below. In particular, the construction of the absorbers does not need any changes. Obviously, the definition of a vortex has to be adapted to the quasirandom setting. More precisely, instead of condition (V4) in Definition 3.6, one now requires that for every set $A \subseteq U_{i-1}$ with $|A| \leq 4$ the common neighbourhood in U_i of the vertices in A has size $(1 \pm \xi)p^{|A|}|U_i|$, where ξ is a small error parameter. This definition ensures that the typicality condition is preserved throughout the iterative absorption procedure. Moreover, it allows to perform the Cover down step for each $i \in [\ell]$. In this step, we now use that for every vertex $x \in U_{i-1}$, the link graph $G[N_G(x) \cap U_i]$ will be $(\sqrt{\xi}, 3, p)$-typical (and thus contains many edge-disjoint perfect matchings), replacing the minimum degree condition in Lemma 3.10(i).

Theorem 1.3 has the advantage that it does not rely on the (unknown) parameter δ^{0+}. Indeed, in this setting, the required approximate decompositions can be obtained using the nibble method, introduced by Rödl [27], which has since had an enormous influence on combinatorics. A collection $\mathcal{T}$ of triangles in a graph G is (ξ, p)-*regular* if every edge of G is contained in $(1 \pm \xi)p^2 n$ triangles. The following follows for instance from a result in [2].

Corollary 4.1. *Let $1/n \ll \xi \ll \gamma, p$. Let G be a graph on n vertices which contains a (ξ, p)-regular collection of triangles. Then G contains a K_3-decomposable subgraph H such that $\Delta(G - H) \leq \gamma n$.*

Clearly, the collection of all triangles of a $(\xi, 2, p)$-typical graph is (ξ, p)-regular, thus we immediately obtain approximate K_3-decompositions in such a graph. However, Corollary 4.1 itself is not sufficient to replace δ^{0+} in the proof of Theorem 1.2. This is because of the requirement $\xi \ll \gamma$. Though this is a reasonable assumption in Corollary 4.1, it would make it impossible to control the error parameter ξ during the iterative absorption procedure, since the parameter γ, which controls the maximum degree of the leftover of the approximate decomposition, feeds (via the Cover down step) into the typicality parameter for the subsequent iteration step. To overcome this, we need an intermediate step which 'boosts' the regularity parameter. The aim is to guarantee approximate decompositions with leftover maximum degree γn in a $(\xi, 4, p)$-typical graph even if $\gamma \ll \xi$. To achieve this, instead of applying Corollary 4.1 to the collection of all triangles of G, we find a suitable subcollection which is (ξ', p)-regular, where $\xi' \ll \gamma$, to which we can then apply Corollary 4.1. The idea is to choose such a collection randomly, according to a suitable probability distribution. Such a probability distribution can be thought of as a 'fractional triangle-equicovering', which is similar to a fractional triangle decomposition. Indeed, our basic tool to find such a fractional triangle-equicovering is a so-called K_5-*shifter*, which was introduced in the context of fractional decompositions in [3]. A K_5-shifter is a 'local' function which allows to adjust the total weight of one edge without affecting the total weight of any other edge. A similar idea of 'regularity boosting' has also been successfully applied e.g. in [1].

Lemma 4.2. *Let $1/n \ll \xi, p$ and $\xi \leq p^7/20$. Any $(\xi, 4, p)$-typical graph on n vertices contains an $(n^{-1/3}, p/2)$-regular collection of triangles.*

Proof. Let G be a $(\xi, 4, p)$-typical graph on n vertices, and let $\mathcal{T}^{(3)}$ be the collection of all triangles in G. Moreover, let $\mathcal{T}^{(5)}$ be the collection of all K_5's in G. For every $e \in E(G)$ and $i \in \{3, 5\}$, let $\mathcal{T}^{(i)}(e)$ denote the set of all elements of $\mathcal{T}^{(i)}$ which contain e as an edge.

Assume, for the moment, that $\psi \colon \mathcal{T}^{(3)} \to [0, 1]$ is a function such that for every edge $e \in E(G)$, we have

$$\sum_{T \in \mathcal{T}^{(3)}(e)} \psi(T) = \frac{1}{4}p^2 n.$$

We can then choose a random subcollection $\mathcal{T}' \subseteq \mathcal{T}^{(3)}$ by including every $T \in \mathcal{T}^{(3)}$ with probability $\psi(T)$, all independently. We then have for every $e \in E(G)$ that the expected number of triangles in $\mathcal{T}'$ containing e is $\frac{1}{4}p^2 n$. Using Lemma 2.1(ii), it is then easy to see that with high probability, $\mathcal{T}'$ is $(n^{-1/3}, p/2)$-regular.

It remains to show that ψ exists. Note that since G is $(\xi, 4, p)$-typical, we have $|\mathcal{T}^{(3)}(e)| = (1 \pm \xi)p^2 n$ and $|\mathcal{T}^{(5)}(e)| = (1 \pm \xi)^3 p^9 n^3/6$ for all $e \in E(G)$. Thus, for all $e \in E(G)$,

$$(4.1) \qquad \text{defining } c_e := \frac{p^2 n - |\mathcal{T}^{(3)}(e)|}{4 \cdot |\mathcal{T}^{(5)}(e)|}, \text{ we have } |c_e| \le \frac{6\xi p^2 n}{4(1-\xi)^3 p^9 n^3} \le \frac{3\xi}{p^7 n^2}.$$

For every $e \in E(G)$ and $J \in \mathcal{T}^{(5)}(e)$, we define a function $\psi_{e,J} \colon \mathcal{T}^{(3)} \to \mathbb{R}$ as follows: for $T \in \mathcal{T}^{(3)}$ with $T \subseteq J$, let

$$(4.2) \qquad \psi_{e,J}(T) := \begin{cases} 1/3, & |V(T) \cap e| \in \{0, 2\}; \\ -1/6, & |V(T) \cap e| = 1; \end{cases}$$

and let $\psi_{e,J}(T) := 0$ if $T \not\subseteq J$. Observe that for all $e' \in E(G)$,

$$(4.3) \qquad \sum_{T \in \mathcal{T}^{(3)}(e')} \psi_{e,J}(T) = \begin{cases} 1, & e' = e, \\ 0, & e' \ne e. \end{cases}$$

We now define $\psi \colon \mathcal{T}^{(3)} \to [0, 1]$ as

$$\psi := \frac{1}{4} + \sum_{e \in E(G)} c_e \sum_{J \in \mathcal{T}^{(5)}(e)} \psi_{e,J}.$$

For every $e \in E(G)$, we have

$$\sum_{T \in \mathcal{T}^{(3)}(e)} \psi(T) = \frac{1}{4}|\mathcal{T}^{(3)}(e)| + \sum_{e' \in E(G)} c_{e'} \sum_{J \in \mathcal{T}^{(5)}(e')} \sum_{T \in \mathcal{T}^{(3)}(e)} \psi_{e',J}(T)$$

$$\overset{(4.3)}{=} \frac{1}{4}|\mathcal{T}^{(3)}(e)| + c_e|\mathcal{T}^{(5)}(e)| \overset{(4.1)}{=} \frac{1}{4}p^2 n,$$

as desired. Moreover, for every $T \in \mathcal{T}^{(3)}$, there are at most $\binom{n}{2}\binom{5}{2} \le 5n^2$ pairs (e, J) for which $e \in E(G)$, $J \in \mathcal{T}^{(5)}(e)$ and $T \subseteq J$. Hence,

$$|\psi(T) - 1/4| \;\le\; \sum_{e \in E(G), J \in \mathcal{T}^{(5)}(e) \colon T \subseteq J} |c_e||\psi_{e,J}(T)| \overset{(4.1),(4.2)}{\le} 5n^2 \cdot \frac{3\xi}{p^7 n^2} \cdot \frac{1}{3} \le 1/4,$$

implying that $0 \le \psi(T) \le 1$ for all $T \in \mathcal{T}^{(3)}$, as needed. $\qquad \square$

5. Concluding remarks: general designs

Instead of working with the typicality notion as in the proof of Theorem 1.3, one can also impose more specific conditions on G that allow the proof to work. For instance, a natural condition is 'regularity', meaning that every edge is contained in roughly the same number of triangles. Another requirement arises from the regularity boosting step, for which we would also require that every edge lies in $\Omega(n^3)$ copies of K_5. These properties can also be formulated for r-uniform hypergraphs, which led to the definition of 'supercomplexes' in [10]. This allowed for the treatment of the minimum degree setting (which corresponds to Theorem 1.2) and the quasirandom setting (which corresponds to Theorem 1.3) within a unified framework.

The proof in [10, 11] proceeds by induction on the uniformity r, both for the construction of exclusive absorbers as well as in the Cover down step. In fact, we implicitly already used induction in the proof of Theorem 1.2: firstly, in the cover down step for triangles, we assumed the existence of a perfect matching in an n-vertex graph of minimum degree at least $n/2$ (note that a perfect matching can be viewed as a $(2, 1, 1)$-design). Secondly, in the simplified construction of absorbers mentioned after the proof of Lemma 3.4, we choose a perfect matching in the link graph of a given vertex. In turn, to prove e.g. the existence of decompositions of hypercliques into tetrahedra (which corresponds to the existence of $(4, 3, 1)$-designs), a strengthening of Theorem 1.3 (as well as the existence of suitable perfect matchings) is used both in the construction of absorbers as well as in the Cover down step.

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