Spanning cycles in random directed graphs

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Abstract

We show that, in almost every n-vertex random directed graph process, a copy of every possible n-vertex oriented cycle will appear strictly before a directed Hamilton cycle does, except of course for the directed cycle itself. Furthermore, given an arbitrary n-vertex oriented cycle, we determine the sharp threshold for its appearance in the binomial random directed graph. These results confirm, in a strong form, a conjecture of Ferber and Long.

1 Introduction

Hamilton cycles in random graphs have been extensively studied since the early work of Erdős and Rényi [6] on random graphs. Improving on seminal work by Pósa [18] and Korshunov [13], the sharp appearance threshold of the Hamilton cycle was determined in 1983 by Bollobás [3], and Komlós and Szemerédi [12], who showed that, if $p = (\log n + \log \log n + \omega(1))/n$, then the binomial random graph $G(n, p)$ is, with high probability, Hamiltonian. If $p = (\log n + \log \log n - \omega(1))/n$, then, with high probability, $G(n, p)$ has a vertex with degree at most 1, and therefore contains no Hamilton cycle. For such ranges of p, then, with high probability, the property of Hamiltonicity in $G(n, p)$ is exactly concurrent with the property of the minimum degree being at least 2.

Such a result can be made more precise by considering the n-vertex *random graph process* G_0, G_1, $\ldots$, $G_{n(n-1)/2}$, where G_0 is the graph with vertex set $[n]$ and no edges, and each graph G_i, $i \geq 1$, in the sequence, is formed from G_{i-1} by adding a new edge taken uniformly at random from the non-edges of G_{i-1}. Independently, Bollobás [4], and Ajtai, Komlós and Szemerédi [1], showed that, in almost every random graph process, the first graph G_i with minimum degree at least 2 is Hamiltonian. Further results on the Hamiltonicity of random graphs, including counting and packing results, can be found in Frieze's comprehensive bibliography [9].

Hamilton cycles have also been extensively studied in random directed graphs (digraphs). Here, a directed Hamilton cycle is a cycle through every vertex of a digraph whose edges are directed in the same direction around the cycle. In 1980, McDiarmid [15] gave a beautiful coupling argument which, when applied to Hamilton cycles, shows that, if $p = (\log n + \log \log n + \omega(1))/n$, then $D(n, p)$ is Hamiltonian with high probability, where $D(n, p)$ is the binomial random digraph with n vertices and edge probability p. This coupling argument is crucial to this paper, and is covered in Sections 2.2 and 3.3. For *directed* Hamiltonicity, the natural local obstruction is that each vertex must have at least one in-neighbour and at least one out-neighbour so that a directed cycle may pass through it. In $D(n, p)$, if $p = (\log n + \omega(1))/n$, then, with high probability, each vertex will have this property, while, if $p = (\log n - \omega(1))/n$, with high probability at least one vertex will not. Similarly to the undirected case, this local obstruction coincides with when we can expect the binomial random digraph to be Hamiltonian, as shown by Frieze [8]. That is, if $p = (\log n + \omega(1))/n$, then, with high probability, $D(n, p)$ is Hamiltonian.

The n-vertex random digraph process $D_0, D_1, \ldots, D_{n(n-1)}$ begins with the digraph D_0 with vertex set $[n]$ and no edges, and each digraph D_i, $i \geq 1$, in the sequence is formed from D_{i-1} by adding a new directed edge taken uniformly at random from the non-edges of D_{i-1}. Frieze [8] gave the corresponding

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result for Hamilton cycles in the random digraph process to that shown in the random graph process. That is, in almost every random digraph process, the first digraph in which every vertex has in- and out-degree at least 1 is Hamiltonian.

The directed n-vertex cycle is the most natural generalisation of the undirected n-vertex cycle, but we may also consider other n-vertex oriented cycles. An oriented cycle is any digraph formed by taking an undirected cycle and orienting its edges. Ferber and Long [7] studied such cycles in the binomial random digraph, and noted that McDiarmid's coupling argument gives that, for any n-vertex oriented cycle C, if $p = (\log n + \log\log n + \omega(1))/n$, then $D(n,p)$ contains a copy of C with high probability. Furthermore, they conjectured that this should be true as long as $p = (\log n + \omega(1))/n$.

The local obstruction to a copy of an n-vertex oriented cycle C in $D(n,p)$ is different depending on the pattern of directions on C. For example, consider the anti-directed Hamilton cycle, where, for even n, the edges change direction at every opportunity around the cycle so that each vertex has in-degree 0 or out-degree 0. If $p = (\log n + 2\log\log n + \omega(1))/2n$, then with high probability every vertex in $D(n,p)$ has out-degree at least 2 or in-degree at least 2, and thus has no local obstruction to the containment of an anti-directed cycle. This is tight up the function $\omega(1)$, and, very recently, Frieze, Pérez-Giménez and Prałat [10] confirmed that this is also when we may expect an anti-directed Hamilton cycle to appear in $D(n,p)$. Thus, compared to the directed Hamilton cycle, we need only around one half of the edge probability to typically find a anti-directed Hamilton cycle. More generally, Frieze, Pérez-Giménez and Prałat [10] studied n-vertex cycles in which the pattern of edges repeats after a fixed interval (with respect to n), and determined which local conditions are likely to imply the existence of such a cycle in the random digraph process. Indeed, they showed that, except for the anti-directed and directed Hamilton cycle, these cycles are likely to appear in the random digraph process as soon as each vertex has total in- and out-degree at least 2.

In this paper, we show that, with high probability, a much larger range of n-vertex oriented cycles will appear in the random digraph process as soon as each vertex has total in- and out-degree at least 2. Our condition on the cycle is only that it has at least $n^{1/2+o(1)}$ vertices where the direction of the edges changes (that is, vertices which have in- or out-degree 0) and at least polylogarithmically many vertices where the direction of the edges is maintained (that is, which have in- and out-degree 1).

Notably, we show that these cycles are likely to appear *simultaneously* at this point in the random digraph process. Determining the threshold for the simultaneous containment of every possible n-vertex oriented cycle in $D(n,p)$ was the original motivation behind this work. For this, we show that, in almost every random digraph process, the first digraph in which every vertex has both in- and out-degree at least 1 contains a copy of every n-vertex oriented cycle. In fact, the directed Hamilton cycle is likely to be strictly the last such cycle to appear. From these results, it follows simply that, if $p = (\log n + \omega(1))/n$, then $D(n,p)$ contains a copy of every n-vertex oriented cycle. In particular, this confirms the conjecture of Ferber and Long [7] stated above. These results are summarised in the following theorem.

Theorem 1.1. *Let $D_0, D_1, \ldots, D_{n(n-1)}$ be the n-vertex random digraph process. Let m_1 be the largest integer m for which $\delta^+(D_m) = 0$ or $\delta^-(D_m) = 0$. Then, with high probability,*

(i) D_{m_1} contains a copy of every n-vertex oriented cycle except for the directed n-vertex cycle, and

(ii) D_{m_1+1} contains a copy of every n-vertex oriented cycle.

Let m_0 be the smallest integer m for which $d^+_{D_m}(v) + d^-_{D_m}(v) \geq 2$ for every $v \in V(D_m)$. Then, with high probability,

(iii) D_{m_0} contains a copy of every n-vertex oriented cycle with at least $n^{1/2}\log^3 n$ changes of direction and at most $n - \log^4 n$ changes of direction.

We will also find, given any n-vertex oriented cycle C, the sharp threshold for the appearance of C in $D(n,p)$, where the thresholds vary from $p = \log n/2n$ to $p = \log n/n$ (see Theorem 1.3). If C has few vertices with in- and out- degree 1, then the random graph must have few vertices with both in- and out-degree exactly 1. If C has few vertices with in-degree 0 or out-degree 0, then the random graph must have few vertices with in-degree 0 or out-degree 0. As p increases from $(1+o(1))\log n/2n$ to $(1+o(1))\log n/n$, the expected number of vertices in D with in- or out-degree 0 decreases from $n^{1/2+o(1)}$ to 0. The expected number of vertices with both in- and out-degree 1 is much smaller, and, as p increases in this interval, it quickly decreases from $n^{o(1)}$ to 0. Thus, for the sharp threshold for C we focus on the vertices in C with in- or out-degree 0. We define p_C below, before showing that this is the sharp threshold in Theorem 1.3.

Definition 1.2. Given an n-vertex oriented cycle C, let $\lambda(C)$ be the number of vertices of C with in- or out-degree 0 in C. If $\lambda(C) = 0$, then let $p_C = \log n/n$, while if $\lambda(C) > 0$, let $p_C = \max\{\log n, 2(\log n - \log \lambda(C))\}/2n$.

Theorem 1.3. *For each $\varepsilon > 0$ and function $p = p(n)$, with high probability, $D(n,p)$ contains a copy of every n-vertex oriented cycle with $p_C \leq (1-\varepsilon)p$ and no copy of any n-vertex oriented cycle with $p_C \geq (1+\varepsilon)p$.*

Both Theorem 1.1 and Theorem 1.3 follow from a stronger theorem, Theorem 2.4, which gives a better indication of where in the random digraph process we can expect an arbitrary spanning oriented cycle to appear. However, there are n-vertex cycles C whose point of appearance cannot be (with high probability) determined only from the evolving degree sequence of the n-vertex random digraph process. For example, consider an n-vertex cycle C with exactly two vertices with out-degree 0 and exactly two vertices with in-degree 0, which are in sequence ℓ vertices apart on the cycle, for some function $\ell = \ell(n)$. With positive probability the last two vertices in the n-vertex random digraph process $D_0, D_1, \ldots, D_{n(n-1)}$ with in- or out-degree 0 will both have in-degree 0. Then, whether a copy of C appears in the first digraph D_j which has at most 2 vertices with out-degree 0 and at most 2 vertices with in-degree 0 can depend, for certain values of $\ell \approx \log n/2\log\log n$ on the different paths in D_j with length 2ℓ between the last two vertices with in-degree 0. Carefully selecting the value of ℓ, we can find a sequence $\ell = \ell(n)$ where the probability a copy of C exists in D_j is bounded away from 0 and 1.

The key new method used by this paper is a combination of constructive techniques along with McDiarmid's coupling. After stating our notation, this is sketched in detail in Section 2, before we state our main technical theorem, Theorem 2.3, and its application to the random digraph process, Theorem 2.4. In Section 3, we prove Theorem 2.3, from which we then deduce Theorem 2.4 in Section 4.

2 Preliminaries

2.1 Notation

A digraph D has vertex set $V(D)$ and edge set $E(D)$, where $E(D)$ is a collection of ordered distinct vertex pairs from $V(D)$. We let $e(D) = |E(D)|$ and $|D| = |V(D)|$. We say that uv is an edge of D if $(u,v) \in E(D)$, and consider this edge as directed from u to v. Where $uv \in E(D)$, we say that v is an out-neighbour (or $+$-neighbour) of u and u is an in-neighbour (or $-$-neighbour) of v. For each $\diamond \in \{+,-\}$, we let $N^\diamond_D(v)$ be the set of $\diamond$-neighbours of v in D, and set $d^\diamond_D(v) = |N^\diamond_D(v)|$. Where it is clear from context, we omit the subscript. We let $\Delta^+(D)$ and $\delta^+(D)$ be the maximum and minimum out-degree of D respectively, and define $\Delta^-(D)$ and $\delta^-(D)$ similarly. Where $\pm$ is used, we mean that the statement holds with $\pm$ replaced by both $+$ and by $-$.

Given $A, B \subset V(D)$, $v \in V(D)$ and $\diamond \in \{+,-\}$, we let $N^\diamond_D(v,B) = N^\diamond_D(v) \cap B$, and $N^\diamond_D(A) = (\cup_{w\in A} N^\diamond_D(w)) \setminus A$ and $N^\diamond_D(A,B) = N^\diamond_D(A) \cap B$. The digraph $D[A]$ is the digraph D induced on the vertex set A. Given an edge e with vertices in $V(D)$, the digraphs $D+e$ and $D-e$ have vertex set $V(D)$ and edge sets $E(D) \cup \{e\}$ and $E(D) \setminus \{e\}$ respectively. Given a vertex set $A \subset V(D)$, the digraph $D - A$ is the digraph $D[V(D) \setminus A]$. We use similar notation for, for example, $D - v$ with $v \in V(D)$, and $D + E$, where E is a set of edges.

Given two digraphs H and G, we say that $H \subsetneq G$ if G contains a copy of H. We denote by $D(n,p)$ the binomial random digraph with vertex set $[n] = \{1, \ldots, n\}$, in which each possible edge uv is included independently at random with probability p. In a digraph D, we say vertices $u, v \in V(D)$ are at least k apart if their graph distance in the underlying undirected graph of D is at least k.

Our notation for graphs is analogous to that defined above for digraphs. We also use standard asymptotic notation such as $O(n)$, $\Theta_C(n)$, where the implicit constant(s) depend on the variable in the subscript, if any. We say an event holds *with high probability* if the probability which with it holds tends to 1 as $n \to \infty$. For each $n \in \mathbb{N}$, we use $\log^{[2]} n = \log\log n$ and $\log^{[3]} n = \log\log\log n$. All logarithms are natural, and we omit rounding signs whenever they are not crucial.

2.2 Proof overview

In our proof sketch we will concentrate on how to show that many different spanning cycles appear simultaneously in the binomial random digraph. Let us say then that $p = \lambda \log n/n$, for some large constant λ, and that we wish to show that $D(n,p)$ contains a copy of every n-vertex oriented cycle with high probability. We note first that a simple union bound is not strong enough for this. Indeed, given an n-vertex cycle C whose edges are oriented with any directions, we have, for $D = D(n,p)$ and an arbitrary $v \in [n]$, that

$$\mathbb{P}(C \subsetneq D(n,p)) \leq \mathbb{P}(d^+_D(v) + d^-_D(v) > 0) = 1 - (1-p)^{2(n-1)} = 1 - \exp(-\Theta_\lambda(\log n)).$$

As there are $2^{(1-o(1))n}$ oriented cycles with length n up to isomorphism, we thus cannot prove a bound on $\mathbb{P}(C \subsetneq D(n,p))$ before taking a union bound over all the n-vertex oriented cycles C.

On the other hand, it would suffice to find some pseudorandom properties which $D(n,p)$ has with high probability and show that any digraph with these properties contains any n-vertex oriented cycle C. However, for general cycles C, this seems to be rather challenging. Instead, we combine the 'union bound' approach and the 'pseudorandom' approach, as follows.

Taking two random digraphs D_0 and D_1, each distributed as $D(n,p/2)$, we define a notion of a 'pseudorandom digraph' and show that

$$\mathbb{P}(D_0 \text{ is pseudorandom}) = 1 - o(1)$$

and, for any n-vertex oriented cycle C,

$$\mathbb{P}(C \subsetneq D_0 \cup D_1 | D_0 \text{ is pseudorandom}) = 1 - \exp(-\omega(n)). \tag{1}$$

Choosing first the random digraph D_0, and then taking a union bound over all cycles C, these statements easily combine (see Section 4.3) to show that

$$\mathbb{P}(D(n,p) \text{ contains every } n\text{-vertex cycle}) \geq \mathbb{P}(D_0 \cup D_1 \text{ contains every } n\text{-vertex cycle}) = 1 - o(1).$$

Instead of proving (1) directly, we now employ McDiarmid's coupling technique (as discussed extensively in Section 3.3). For this, consider the following random digraph, $D^*(n,q)$.

Definition 2.1. Let $D^*(n,q)$ be the random digraph with vertex set $[n]$ where each pair of edges uv and vu are included together independently at random with probability q, and otherwise excluded.

Let D^*_1 be distributed as $D^*(n,p/2)$. A simple use of McDiarmid's coupling technique (see Section 3.3) shows that

$$\mathbb{P}(C \subsetneq D_0 \cup D_1 | D_0 \text{ is pseudorandom}) \geq \mathbb{P}(C \subsetneq D_0 \cup D^*_1 | D_0 \text{ is pseudorandom}).$$

Therefore, to prove (1), it is sufficient to show that

$$\mathbb{P}(C \subsetneq D_0 \cup D^*_1 | D_0 \text{ is pseudorandom}) = 1 - \exp(-\omega(n)). \tag{2}$$

Now, we observe that D^*_1 has the same distribution as the random graph $G(n,p/2)$ with each edge uv replaced by the two directed edges uv and vu. This allows us to use (undirected) graph techniques to find paths and cycles in the underlying graph of D^*_1, safe in the knowledge that such a path or cycle will exist in D^*_1 with any orientations on its edges. This is the benefit of working with D^*_1. However, for (2), we can only use graph techniques which use properties of $G(n,p/2)$ which hold with probability $1 - \exp(-\omega(n))$.

Key here is that one part of the standard proof of the Hamiltonicity of $G(n,p)$ uses a 'sprinkling' technique that works with probability $1 - \exp(-\omega(n))$. So that we may recall this, let G_0 and G_1 be independent random graphs, each distributed as $G(n,p/2)$. Typically, following the approach pioneered by Pósa [18], we show that G_0 is likely to be an 'expander' (see Section 3.1), and then, given that G_0 is an 'expander', that $G_0 \cup G_1$ is likely to contain a Hamilton cycle. In fact, we have, where C_n is an n-vertex cycle, that

$$\mathbb{P}(C_n \subsetneq G_0 \cup G_1 | G_0 \text{ is an 'expander'}) = 1 - \exp(-\omega(n)). \tag{3}$$

This is the undirected version of (2), our version of which we prove as Lemma 3.6.

Unfortunately, the proof of (3) (using the extension-rotation method as given in Section 3.8) cannot be applied to directed graphs to get an arbitrary cycle. Instead, given a pseudorandom digraph D_0, we split D_1^* into two random digraphs D_2^* and D_3^* with equal edge probability and proceed with the following 4 steps.

A We reveal D_2^* to (with very high probability) identify a set of 'bad' vertices $B \subset [n]$ which are hard to cover by paths or cycles in D_2^*.

B We use D_0 to find sections of the cycle C covering these bad vertices, where the sections have endvertices in $[n] \setminus B$.

C We connect these sections using D_2^* into a single section of the cycle (using that the endvertices are not 'bad'). Say the path found in $D_0 \cup D_2^*$ is a path P with endvertices x and y. When we do this, we ensure that $D_2^* - V(P - x - y)$ is an 'expander'.

D Using our version of (3), we reveal D_3^* and show that, given $D_2^* - V(P - x - y)$ is an 'expander', an x, y-path through every vertex in $(D_2^* \cup D_3^*) - V(P - x - y)$ exists with very high probability.

These steps are given in more detail in Section 3. We next give our definition of pseudorandomness and the statement of our main technical theorem, before discussing how it can be applied to the random digraph process.

2.3 Pseudorandomness and our main technical theorem

For our definition of pseudorandomness, we take the simplest conditions we need for our methods to work. We require our pseudorandom digraph to satisfy some maximum in- and out-degree condition (**A1** below), some minimum in- and out-degree condition (**A2** below), and a condition that gives rise to some digraph 'expansion' (**A3** below). Additionally, the pseudorandom digraph D has an *exceptional set* of vertices X, which will arise from low in- or out-degree vertices in the random digraph process.

Definition 2.2. Given an n-vertex digraph D and a vertex set $X \subset V(D)$, D is *pseudorandom with exceptional set* X if the following hold.

A1 $\Delta^{\pm}(D) \leq 100 \log n$.

A2 For each $v \in V(D)$, $d^{\pm}(v, V(D) \setminus X) \geq \log n/500$.

A3 For any sets $A, B \subset V(D)$ and $\diamond \in \{-, +\}$, with $|A| \leq n \log\log n/\log n$, and, for each $v \in A$, $d^{\diamond}(v, B) \geq (\log n)^{2/3}$, we have $|B| \geq |A|(\log n)^{1/3}$.

We wish to apply our methods to digraphs in the n-vertex random digraph process which have minimum in- or out-degree strictly less than $\log n/500$. As **A2** will not hold, such a digraph is not pseudorandom. However, as discussed below, we will modify such a digraph into a pseudorandom digraph. Once we find a spanning cycle, we will undo this modification, and therefore the spanning cycle we find will need to have some additional properties. This motivates our main theorem, Theorem 2.3, where prespecified vertices in the cycle are copied to prespecified vertices in the exceptional set X.

Theorem 2.3. *There is some n_0 such that the following holds for each $n \geq n_0$. Let D_0 be an n-vertex digraph containing $X \subset V(D_0)$, with $|X| \leq n^{3/4}$, so that D_0 is pseudorandom with exceptional set X. Let C be an n-vertex oriented cycle and let $P \subset C$ be a path with length at most $n/10$. Let $f : X \to V(P)$ be an injection such that the vertices in $f(X)$ are pairwise at least $20 \log n/\log\log n$ apart on P, and let $D_1 = D(n, \log n/10^3 n)$*

Then, with probability at least $1 - 2\exp(-2n)$, $D_0 \cup D_1$ contains a copy of C in which $f(x)$ is copied to x for each $x \in X$.

We prove this theorem with the strategy outlined in Section 2.2. To apply it to the random digraph process, we first modify the random digraphs. This is explained in more detail in Section 4, but, roughly, before Theorem 2.3 is applied, we first use a conditioning argument (from Krivelevich, Lubetzky and Sudakov [14]) to reserve some random edges to act as D_1 in Theorem 2.3, and then identify the low in- or out-degree vertices in a digraph in the random digraph process. Assigning each low degree vertex two neighbours, we embed well-spaced paths of length 2 from the cycle to these vertices and their chosen neighbours. Replacing each embedded path of length 2 by a new vertex and modifying the cycle and digraph from the random digraph process, we apply Theorem 2.3 using the random edges we have reserved as D_1, gathering the new vertices into the exceptional set X. Altogether, this will give us the following theorem.

Theorem 2.4. *Let $D_0, D_1, \dots, D_{n(n-1)}$ be the n-vertex random digraph process. For each i with $0 \leq i \leq n(n-1)$, let s_i be the number of vertices in D_i with in-degree or out-degree 0 and let t_i be the number of vertices in D_i with in-degree 1 and out-degree 1.*

Then, with high probability, the following holds for each $i \in \{0, 1, \dots, n(n-1)\}$. If $d^+_{D_i}(v) + d^-_{D_i}(v) \geq 2$ for each $v \in V(D_i)$, then D_i contains a copy of every n-vertex oriented cycle with at least $1 + (s_i - 1)\log n$ changes in direction and at most $n - 1 - (t_i - 1)\log n$ changes in direction.

Theorem 1.1 and Theorem 1.3 follow straight-forwardly from Theorem 2.4. We show this in detail in Section 4.3, following the proof of Theorem 2.4. In Section 3, we prove Theorem 2.3.

3 Proof of Theorem 2.3

In Section 3.1 we state our component results, before combining them to prove Theorem 2.3 in Section 3.2. We prove these component results in Sections 3.3–3.8.

3.1 Components of the proof of Theorem 2.3

The following component parts are contextualised in the proof guide in Section 2.2.

3.1.1 Coupling argument. We prove the following implication of McDiarmid's coupling argument in Section 3.3, where an oriented graph is a directed graph in which there is at most one edge between any pair of vertices.

Theorem 3.1. *Let $p \in [0,1]$ and $n \in \mathbb{N}$. Let $\mathcal{H}$ be a set of oriented graphs with vertex set $[n]$, let D_0 be a digraph with vertex set $[n]$, let $D_1 = D(n,p)$ and let $D_1^* = D^*(n,p)$.*

Then, $\mathbb{P}(\exists H \in \mathcal{H} : H \subset D_0 \cup D_1) \geq \mathbb{P}(\exists H \in \mathcal{H} : H \subset D_0 \cup D_1^*)$.

3.1.2 Partitioning lemma. We partition the vertex set of our pseudorandom digraph for the steps outlined in Section 2.2 using the following lemma, which is proved in Section 3.4.

Lemma 3.2. *There is some n_0 such that the following holds for each $n \geq n_0$. Suppose D_0 is an n-vertex pseudorandom digraph with exceptional set $X \subset V(D_0)$ satisfying $|X| \leq n^{3/4}$.*

Then, there is a partition $V_0 \cup V_1 \cup V_2$ of $V(D_0) \setminus X$ with $|V_1| = |V_2| = \lfloor n/4 \rfloor$ such that, for each $v \in V(D_0)$ and $i \in [2]$, we have $d^{\pm}(v, V_i) \geq \log n/5000$.

3.1.3 Steps A and C. The next lemma, Lemma 3.4, carries out Steps A and C in Section 2.2, and is proved in Sections 3.5 and 3.6. It identifies the set of 'bad' vertices B for Step A, while showing the connectivity property that is then used for Step C. We state it after formalising how we need 'small sets to expand', as follows.

Definition 3.3. An n-vertex graph G is a *10-expander* if it is connected and, given any subset $A \subset V(G)$ with $|A| \leq n/20$, we have $|N(A)| \geq 10|A|$.

Lemma 3.4. *There is some n_0 such that the following holds for each $n \geq n_0$. Let $V_0 \subset [n]$ satisfy $|V_0| \geq n/4$, and let $p = \log n/10^4 n$ and $G = G(n,p)$. Then, with probability at least $1 - \exp(-2n)$, G contains a set $B \subset [n]$ with $1 \leq |B| \leq n \log^{[3]} n/\log n$ and the following property.*

Suppose we have any $k \geq 1$, and any integers $\ell_i \geq 10 \log n/\log^{[2]} n$, $i \in [k]$, such that $\sum_{i\in[k]} \ell_i \leq n/8$, and distinct vertices $x_1, \ldots, x_k, y_1, \ldots, y_k \in V(G) \setminus (V_0 \cup B)$. Then, there is a set of internally vertex-disjoint paths $P_1, \ldots, P_k$ in G such that the following hold with $V_1 = V_0 \setminus (B \cup (\cup_{i\in[k]} V(P_i)))$.

- *For each $i \in [k]$, P_i is an x_i, y_i-path in G with length ℓ_i and internal vertices in $V_0 \setminus B$.*
- *For each $A \subset V(G)$ with $V_1 \subset A$ and $A \cap B = \emptyset$, $G[A]$ is a 10-expander.*

3.1.4 Step B. The next lemma, Lemma 3.5, carries out Step B, and is proved in Section 3.7. The lemma will be applied to the exceptional set X of the pseudorandom digraph D_0 and the set of 'bad' vertices B from Lemma 3.4 (split as $B = B^+ \cup B^-$). We use it to embed paths from the cycle to cover $X \cup B$, with vertices in X embedded to prespecified vertices, so that these paths have endvertices outside of $X \cup B$ (so that their endvertices are 'good' vertices).

Lemma 3.5. *There is some n_0 such that the following holds for each $n \geq n_0$. Let D be an n-vertex pseudorandom digraph with exceptional set X. Let B^+, B^-, A^+, A^- be disjoint sets in $V(D) \setminus X$, and suppose that*

- *for each $v \in V(D)$ and $\diamond \in \{+,-\}$, $d^\diamond(v, B^\diamond \cup A^\diamond) \geq \log n/5000$, and*
- *$|X|, |B^+|, |B^-| \leq n \log^{[3]} n/\log n$.*

Let $B = B^+ \cup B^-$. Let $k = |X \cup B|$, and let $\{P_i : i \in [k]\}$ be a collection of vertex-disjoint oriented paths, each with length $2\lceil 4 \log n/\log\log n\rceil$. Let x_i be the midpoint of P_i for each $i \in [k]$, and let $f : X \cup B \to [k]$ be a bijection.

Then, there is some $\bar{B} \subset B$ and a collection of vertex-disjoint oriented paths $\{Q_v : v \in X \cup \bar{B}\}$ such that,

B1 *for each $v \in X \cup \bar{B}$, Q_v is a copy in D of a portion of $P_{f(v)}$ with endvertices in $A^+ \cup A^-$ and interior vertices in $X \cup B$, in which $x_{f(v)}$ is copied to v, and*

B2 *$X \cup B$ is contained in $\cup_{v\in X\cup\bar{B}} V(Q_v)$.*

3.1.5 Step D. Finally, the following lemma is used for Step D and is proved in Section 3.8.

Lemma 3.6. *There is some n_0 such that the following holds for each $n \geq n_0$. Let G_0 be a 10-expander with vertex set $[n]$ and let $x, y \in V(G_0)$ be distinct. Let $p = \log n/10^5 n$ and $G_1 = G(n,p)$. Then, with probability at least $1 - \exp(-4n)$, $G_0 \cup G_1$ contains a Hamilton x,y-path.*

3.2 Proof of Theorem 2.3

We now put these component parts together to prove Theorem 2.3, as follows.

Proof of Theorem 2.3. Let n_0 be sufficiently large that each property in Lemmas 3.2, 3.4, 3.5 and 3.6 holds for each $n \geq n_0/2$, and further simple inequalities involving $n \geq n_0$ hold as used below. We will show the property in Theorem 2.3 holds for each $n \geq n_0$. For this, let D_0 be an n-vertex pseudorandom digraph with exceptional set $X \subset V(D_0) = [n]$ such that $|X| \leq n^{3/4}$. Let C be an n-vertex oriented cycle. Let $P \subset C$ be a path with length at most $n/100$. Let $f : X \to V(P)$ be an injection, so that the vertices in $f(X)$ are pairwise at least $100 \log n/\log\log n$ apart on P.

Let $\mathcal{H}$ be the set of copies of C with vertex set $[n]$ in which $f(x)$ is copied to x for each $x \in X$, and let $p = \log n/10^4 n$. Independently, let D_1^*, D_2^*, D_1, and D^* be distributed as $D^*(n,p)$, $D^*(n,p)$, $D(n,10p)$, and $D^*(n,10p)$ respectively. Noting that each pair of edges $\{uv, vu\}$ appears in $D_1^* \cup D_2^*$ independently at random with probability $1-(1-p)^2 \leq 10p$, Theorem 3.1 implies that

$$\mathbb{P}(\exists H \in \mathcal{H} : H \subset D_0 \cup D_1) \geq \mathbb{P}(\exists H \in \mathcal{H} : H \subset D_0 \cup D^*) \geq \mathbb{P}(\exists H \in \mathcal{H} : H \subset D_0 \cup D_1^* \cup D_2^*).$$

Thus, to prove Theorem 2.3, it is sufficient to show that $\mathbb{P}(\exists H\in\mathcal{H}: H\subset D_0\cup D_1^*\cup D_2^*)\geq 1-2\exp(-2n)$.

First, using the property from Lemma 3.2, find a partition $V(D_0)\setminus X=V_0\cup V_1\cup V_2$ with $|V_1|=|V_2|=\lfloor n/4\rfloor$ such that, for each $v\in V(D_0)$ and $i\in[2]$, we have $d^{\pm}(v,V_i)\geq \log n/5000$. Note that $|V_0|\geq n/2-|X|\geq n/4$.

Let G_1 be the underlying undirected graph of D_1^*, noting that G_1 has the same distribution as $G(n,p)$. By the property from Lemma 3.4, with probability at least $1-\exp(-2n)$, there exists a set $B\subset V(G_1)$ with $1\leq |B|\leq n\log^{[3]}n/\log n$ such that the following holds.

C1 For any $k\geq 1$, and any integers $\ell_i\geq 10\log n/\log^{[2]}n$, $i\in[k]$, such that $\sum_{i\in[k]}\ell_i\leq n/8$, and distinct vertices $x_1,\ldots,x_k,y_1,\ldots,y_k\in V(G_1)\setminus(V_0\cup B)$, there is a set of internally vertex-disjoint paths $R_1,\ldots,R_k$ such that the following hold with $\bar{V}_0=V_0\setminus(B\cup(\cup_{i\in[k]}V(R_i)))$.

- For each $i\in[k]$, R_i is an x_i,y_i-path in G_1 with length ℓ_i and internal vertices in V_0.
- For each $A\subset V(G_1)$ with $\bar{V}_0\subset A$ and $A\cap B=\emptyset$, $G_1[A]$ is a 10-expander.

Now, let $k'=|X\cup B|\geq 1$ and note that $k'\leq 2n\log^{[3]}n/\log n$. Let $A^+=V_1\setminus B$, $A^-=V_2\setminus B$, $B^+=B\cap V_1$ and $B^-=B\cap V_2$. Let $\ell=2\lceil 4\log n/\log^{[2]}n\rceil$ and let $\ell_0=10\log n/\log^{[2]}n$. Let P' be the subpath of C containing P which contains ℓ extra vertices on each side. Take paths P_i, $i\in[k']$, in P' which are pairwise a distance at least ℓ_0 apart in C, which each have length 2ℓ, and such that, for each $x\in X$, there is some $j\in[k']$ for which $f(x)$ is the center vertex of P_j. Note that this is possible as $k'(2\ell+2\ell_0)=o(n)$.

By the property from Lemma 3.5, there is some $k\leq k'$ and a set of vertex-disjoint paths Q_i, $i\in[k]$, in D_0, such that the following hold.

C2 For each $i\in[k]$, Q_i is a copy of a portion of P_i with endvertices in $A^+\cup A^-$ and interior vertices in $X\cup B$.

C3 $X\cup B\subset\cup_{i\in[k]}V(Q_i)$.

C4 For each $x\in X$, for some $j\in[k]$, Q_j is the copy of a portion of P' containing x in which $f(x)$ is copied to x.

Note that we can assume, by deleting paths if necessary, that each path Q_i, $i\in[k]$, contains some vertex in $X\cup B$, and hence, by **C2**, has length at least 2. As $k'=|X\cup B|\geq 1$, we have $k\geq 1$.

Pick an arbitrary clockwise direction on C. Relabelling, if necessary, assume that the paths $P_1,\ldots,P_k$ occur on C in clockwise order. For each $i\in[k-1]$, let $\ell_i\geq\ell_0$ be the length of the path between the preimage of Q_i and the preimage of Q_{i+1} on P. For each $i\in[k]$, label the endvertices of Q_i so that Q_i is an x_i,y_i-path and the preimage of x_i occurs earlier in P_i than the preimage of y_i under the clockwise order. Note that, by the choice of the paths P_i, $i\in[k]$, we have that $\sum_{i\in[k]}\ell_i\leq |P|+2\ell\leq n/8$ and, for each $i\in[k-1]$, $\ell_i\geq\ell_0=10\log n/\log^{[2]}n$. Furthermore, the vertices $x_1,\ldots,x_k,y_1,\ldots,y_k$ are all distinct as they are endvertices of vertex-disjoint paths with length at least 2, and, by **C2**, these vertices are all in $A^+\cup A^-=(V_1\cup V_2)\setminus B=V(D_0)\setminus(V_0\cup B)$.

By **C1**, we can find paths R_i, $i\in[k-1]$, which are internally vertex-disjoint such that, for each $i\in[k-1]$, R_i is a y_i,x_{i+1}-path in G_1 with length ℓ_i and internal vertices in V_0, and such that, setting $\bar{V}_0=V_0\setminus(B\cup(\cup_{i\in[k-1]}V(R_i)))$, the following holds.

C5 For each $A\subset V(G_1)$ with $\bar{V}_0\subset A$ and $A\cap B=\emptyset$, $G_1[A]$ is a 10-expander.

Now, for each R_i, $i\in[k-1]$, let R_i' be the digraph formed by replacing each edge uv of R_i by both uv and vu. Observe that $(\cup_{i\in[k]}Q_i)\cup(\cup_{i\in[k-1]}R_i')\subset D_0\cup D_1^*$ contains an oriented x_1,y_k-path, Q say, of a portion of P', P'' say, in which the following hold.

- By **C2**, the paths Q_i, $i\in[k]$, have vertices in $A^+\cup A^-\cup(X\cup B)=V_1\cup V_2\cup X\cup B$, and hence, by the definition of $\bar{V}_0$, we have $\bar{V}_0\subset V(G_1)\setminus V(Q)$.
- By **C3**, we have $X\cup B\subset V(Q)$, and, hence, by **C5**, $G_1':=G_1-(V(Q)\setminus\{x_1,y_k\})$ is a 10-expander.
- By **C4**, for each $x\in X$, $f(x)\in V(P'')$ is copied to $x\in V(Q)$.

Let $m = |G_1'|$, and note that $m \geq n - |Q| \geq n - |P''| \geq n/2 \geq n_0/2$. Let G_2 be the underlying graph of $D_2^*[V(G_1')]$ and note that G_2 has the same distribution as $G(m,p)$. Then, by the property from Lemma 3.6, with probability at least $1 - \exp(-4m)$, $G_1' \cup G_2$ contains a Hamilton y_k, x_1-path, S say. Let S^* be S with each edge replaced by a directed edge in each direction, and note that, as $S \subset G_1' \cup G_2$, we have that $S^* \subset D_1^* \cup D_2^*$. Finally, note that a copy of C lies in $Q \cup S^* \subset D_0 \cup D_1^* \cup D_2^*$, in which $f(x)$ is copied to x for each $x \in X$. Therefore, in total, we have found a copy of C in $D_0 \cup D_1^* \cup D_2^*$ in which $f(x)$ is copied to x for each $x \in X$ with probability at least $1 - \exp(-2n) - \exp(-4m) \geq 1 - 2\exp(-2n)$, as required. □

3.3 Coupling argument

To recap, for Theorem 3.1, we have the following situation. We have $p \in [0,1]$, $n \in \mathbb{N}$, and a set $\mathcal{H}$ of oriented graphs with vertex set $[n]$. We have a digraph D_0 with vertex set $[n]$, and random digraphs $D_1 = D(n,p)$ and $D_1^* = D^*(n,p)$. We want to show that

$$\mathbb{P}(\exists H \in \mathcal{H} : H \subset D_0 \cup D_1) \geq \mathbb{P}(\exists H \in \mathcal{H} : H \subset D_0 \cup D_1^*).$$

To do this, we follow closely the approach of McDiarmid [15]. We construct a sequence of random digraphs, denoted $\hat{D}_0, \hat{D}_1, \dots, \hat{D}_{n(n-1)/2}$, beginning with the random digraph $\hat{D}_0$ which has same distribution as D_1^*. Given an arbitrary order of the $n(n-1)/2$ vertex pairs from $[n]$, throughout this sequence of digraphs we steadily decouple each pair of edges uv, vu in D_1^* so that more of these edge pairs appear independently of each other. At the end of this sequence, we have decoupled the appearance of edge pairs in D_1^* until the last digraph in the sequence, $\hat{D}_{n(n-1)/2}$, has the same distribution as D_1. Once this sequence of random digraphs is constructed, we show that, as i increases, some digraph in $\mathcal{H}$ is increasingly likely to appear in $D_0 \cup D_i^*$ (see Claim 1, below). The only change from the proof used by McDiarmid [15] is the introduction of a fixed graph, D_0, whose edges are added to every random graph in this sequence, but this introduces no additional complication.

Proof of Theorem 3.1. Let $\ell = n(n-1)/2$ and enumerate $[n]^{(2)}$ as $e_1 = \{x_1, y_1\}, \dots, e_\ell = \{x_\ell, y_\ell\}$. For each $0 \leq i \leq \ell$, let X_i, Y_i and Z_i be independent Bernoulli random variables which are 1 with probability p, and 0 otherwise. For each $0 \leq j \leq \ell$, let $\hat{D}_j$ be the random digraph with vertex set $[n]$ and edge set

$$\{x_iy_i : 1 \leq i \leq j \text{ and } X_i = 1\} \cup \{y_ix_i : 1 \leq i \leq j \text{ and } Y_i = 1\} \cup \{x_iy_i, y_ix_i : j < i \leq \ell \text{ and } Z_i = 1\}.$$

Note the following.

D1 $\hat{D}_0$ has the same distribution as $D^*(n,p)$, and hence D_1^*.

D2 $\hat{D}_\ell$ has the same distribution as $D(n,p)$, and hence D_1.

D3 For each $j \in [\ell]$, $E(\hat{D}_{j-1}) \triangle E(\hat{D}_j) \subset \{x_jy_j, y_jx_j\}$.

We will show the following claim.

Claim 1. For each $i \in [\ell]$, $\mathbb{P}(\exists H \in \mathcal{H} : H \subset D_0 \cup \hat{D}_i) \geq \mathbb{P}(\exists H \in \mathcal{H} : H \subset D_0 \cup \hat{D}_{i-1})$.

This is sufficient to prove the lemma. Indeed, it follows from Claim 1 that

$$\begin{aligned}\mathbb{P}(\exists H \in \mathcal{H} : H \subset D_0 \cup D_1) &\overset{\mathbf{D2}}{=} \mathbb{P}(\exists H \in \mathcal{H} : H \subset D_0 \cup \hat{D}_\ell) \geq \mathbb{P}(\exists H \in \mathcal{H} : H \subset D_0 \cup \hat{D}_{\ell-1}) \\ &\geq \dots \geq \mathbb{P}(\exists H \in \mathcal{H} : H \subset D_0 \cup \hat{D}_0) \overset{\mathbf{D1}}{=} \mathbb{P}(\exists H \in \mathcal{H} : H \subset D_0 \cup D_1^*).\end{aligned}$$

Thus, $\mathbb{P}(\exists H \in \mathcal{H} : H \subset D_0 \cup D_1) \geq \mathbb{P}(\exists H \in \mathcal{H} : H \subset D_0 \cup D_1^*)$, as required. It is left then only to prove Claim 1.

Proof of Claim 1. Fix an arbitrary $i \in [\ell]$. Let $D_i' = \hat{D}_i - \{x_iy_i, y_ix_i\}$, so that, by **D3**, we also have $D_i' = \hat{D}_{i-1} - \{x_iy_i, y_ix_i\}$. Let $\mathcal{D}$ be the set of possible outcomes for D_i' and fix an arbitrary $D \in \mathcal{D}$. Consider the following three possible cases.

a : $D_0 \cup D$ contains some $H \in \mathcal{H}$.

b : $(D_0 \cup D) + \{x_iy_i, y_ix_i\}$ contains no $H \in \mathcal{H}$.

c : $D_0 \cup D$ contains no $H \in \mathcal{H}$ but $(D_0 \cup D) + \{x_iy_i, y_ix_i\}$ contains some $H \in \mathcal{H}$.

If case **a** occurs, then

$$\mathbb{P}(\exists H \in \mathcal{H} : H \subset D_0 \cup \hat{D}_{i-1} | D'_i = D) = 1 = \mathbb{P}(\exists H \in \mathcal{H} : H \subset D_0 \cup \hat{D}_i | D'_i = D).$$

If case **b** occurs, then

$$\mathbb{P}(\exists H \in \mathcal{H} : H \subset D_0 \cup \hat{D}_{i-1} | D'_i = D) = 0 = \mathbb{P}(\exists H \in \mathcal{H} : H \subset D_0 \cup \hat{D}_i | D'_i = D).$$

If case **c** occurs, then, as $\mathcal{H}$ is a set of oriented graphs, at least one of $D_0 \cup D + x_iy_i$ or $D_0 \cup D + y_ix_i$ contains some $H \in \mathcal{H}$. Thus,

$$\mathbb{P}(\exists H \in \mathcal{H} : H \subset D_0 \cup \hat{D}_i | D'_i = D) \geq p = \mathbb{P}(\exists H \in \mathcal{H} : H \subset D_0 \cup \hat{D}_{i-1} | D'_i = D).$$

Hence, in all cases, and thus for all $D \in \mathcal{D}$, we have

$$\mathbb{P}(\exists H \in \mathcal{H} : H \subset D_0 \cup \hat{D}_i | D'_i = D) \geq \mathbb{P}(\exists H \in \mathcal{H} : H \subset D_0 \cup \hat{D}_{i-1} | D'_i = D).$$

Therefore,

$$\begin{aligned}
\mathbb{P}(\exists H \in \mathcal{H} : H \subset D_0 \cup \hat{D}_i) &= \sum_{D \in \mathcal{D}} \mathbb{P}(D'_i = D) \cdot \mathbb{P}(\exists H \in \mathcal{H} : H \subset D_0 \cup \hat{D}_i | D'_i = D) \\
&\geq \sum_{D \in \mathcal{D}} \mathbb{P}(D'_i = D) \cdot \mathbb{P}(\exists H \in \mathcal{H} : H \subset D_0 \cup \hat{D}_{i-1} | D'_i = D) \\
&= \mathbb{P}(\exists H \in \mathcal{H} : H \subset D_0 \cup \hat{D}_{i-1}).
\end{aligned}$$

□□

3.4 Splitting sets with the local lemma

We will prove Lemma 3.2 with a standard application of the following version of the Local Lemma, due to Lovász (see [19, Theorem 1.1]), where the dependence graph of a set of events $A_1, \ldots, A_n$ is the graph with vertex set $\{A_1, \ldots, A_n\}$ and an edge between A_i and A_j, $i, j \in [n]$ and $i \neq j$, exactly when the events A_i and A_j are not independent.

Theorem 3.7. *Let $A_1, \ldots, A_n$ be events in a probability space Ω with dependence graph G. Suppose there exist $0 < q_1, \ldots, q_n < 1$ such that, for each $i \in [n]$,*

$$\mathbb{P}(A_i) \leq q_i \prod_{j : A_iA_j \in E(G)} (1 - q_j).$$

Then, with strictly positive probability, no such event A_i occurs.

We will also use the following well-known Chernoff bound (see, for example, [5, Corollary 2.3]).

Lemma 3.8. *If X is a binomial variable with standard parameters n and p, denoted $X = \mathrm{Bin}(n, p)$, and ε satisfies $0 < \varepsilon \leq 3/2$, then $\mathbb{P}(|X - \mathbb{E}X| \geq \varepsilon \mathbb{E}X) \leq 2 \exp\left(-\varepsilon^2 \mathbb{E}X/3\right)$.*

Proof of Lemma 3.2. Let $p = 1/5$ and $n \geq 10^3$. Let D be a pseudorandom digraph with vertex set $[n]$ and exceptional set $X \subset V(D)$ which satisfies $|X| \leq n^{3/4}$. Take a partition $V(D) \setminus X = U_0 \cup U_1 \cup U_2$ so that each $v \in V(D) \setminus X$ appears independently in U_0, U_1 and U_2 with probability $1 - 2p$, p and p, respectively. For each $v \in [n]$, let E_v be the event that, for some $i \in [2]$ and $\diamond \in \{+, -\}$, we have $d^{\diamond}(v, U_i) < \log n/5000$. Let E_0 be the event that either $|U_1| > n/4$ or $|U_2| > n/4$.

If E_0 does not hold, then, using that $|X| \leq n^{3/4} \leq n/4$, take a partition $V(D) \setminus X = V_0 \cup V_1 \cup V_2$ such that $U_1 \subset V_1$, $U_2 \subset V_2$, and $|V_1| = |V_2| = \lfloor n/4 \rfloor$. If E_0 does hold then let $V_i = U_i$ for each $i \in [3]$. Note that, if E_0 does not hold, and no E_v, $v \in [n]$, holds, then $V_0 \cup V_1 \cup V_2$ satisfies the requirements in the lemma. Therefore, it is sufficient to show that there is some n_0 such that, whenever $n \geq n_0$, no such event holds with positive probability.

Let $\Delta = 100 \log n$, so that, by **A1** in the definition of pseudorandomness, $\Delta^{\pm}(D) \leq \Delta$. Note that, for each $v \in [n]$, each event E_v has some dependence on only E_0 and the events E_u, $u \in Y_v := \{u \in [n] : (N^+(u) \cup N^-(u)) \cap (N^+(v) \cup N^-(v)) \neq \emptyset\}$, and, furthermore, that $|Y_v| \leq 4\Delta^2$. Let $q_0 = 1/2$ and $q = \exp(-\sqrt{\log n})$. Thus, the lemma follows from the following claim and Theorem 3.7 applied with q_0, and $q_v = q$ for each $v \in [n]$.

Claim 2. There is some n_0 such that, if $n \geq n_0$, then $\mathbb{P}(E_0) \leq q_0(1-q)^n$, and, for each $v \in [n]$, $\mathbb{P}(E_v) \leq q(1-q_0)(1-q)^{4\Delta^2}$.

Proof of Claim 2. First, take an arbitrary $v \in [n]$. By **A2** we have $d^{\pm}(v, V(D) \setminus X) \geq \log n/500$. For each $i \in [2]$, then, $\mathbb{E}|N^{\pm}(v) \cap U_i| \geq \log n/2500$. Therefore, by Lemma 3.8 with $\varepsilon = 1/2$, we have $\mathbb{P}(E_v) = \exp(-\Omega(\log n))$. Noting further that $\Delta^2 q = o(1)$, we have

$$\mathbb{P}(E_v) = \exp(-\Omega(\log n)) = o(q) = o(q(1-4\Delta^2 q)) = o(q(1-q_0)(1-q)^{4\Delta^2}).$$

Thus, for sufficiently large n, $\mathbb{P}(E_v) \leq q(1-q_0)(1-q)^{4\Delta^2}$.

Now, as $|[n] \setminus X| \geq n - n^{3/4} \geq 3n/4$, we have $n/10 \leq \mathbb{E}|U_i| \leq n/5$ for each $i \in [2]$. Thus, by Lemma 3.8 with $\varepsilon = 1/100$, we have $\mathbb{P}(E_0) = \exp(-\Omega(n))$. Therefore, for sufficiently large n, $q_0(1-q)^n \geq \exp(-2qn)/2 = \exp(-o(n)) \geq \mathbb{P}(E_0)$, as required. □ □

3.5 Expansion with very high probability

In this section, we show how to get an expansion property in some subgraph of a random graph with very high probability, which we then use in Section 3.6 to prove Lemma 3.4. We start with the following simple proposition concerning the neighbourhoods of large sets in a random graph.

Proposition 3.9. *For each fixed $c > 0$, if $p = c \log n/n$, then, with probability $1 - \exp(-\omega(n))$, the following hold in $G = G(n,p)$.*

- **E1** *Every set $U \subset V(G)$ with $|U| = n \log^{[3]} n/2 \log n$ satisfies $|N(U)| \geq 9n/10$.*
- **E2** *Every disjoint pair $A, B \subset V(G)$ of subsets of size at least $n/\log^{2/5} n$ have some edge between them.*

Proof. Given any disjoint subsets $U, U' \subset V(G)$ with $|U| = n \log^{[3]} n/2 \log n$ and $|U'| \geq n/100$, the probability there are no edges between U and U' in G is

$$(1-p)^{|U||U'|} = \exp(-\Omega(pn^2 \log^{[3]} n/\log n)) = \exp(-\Omega(n \log^{[3]} n)).$$

Therefore, as there are at most 2^{2n} such pairs $U, U' \subset V(G)$, there are no such pairs with no edges between them with probability at least $1 - 4^n \cdot \exp(-\Omega(n \log^{[3]} n)) = 1 - \exp(-\omega(n))$. If such a property holds, then, for any $U \subset V(G)$ with $|U| = n \log^{[3]} n/2 \log n$, we have $|V(G) \setminus (N(U) \cup U)| < n/100$, and hence

$$|N(U)| \geq n - n/100 - |U| \geq 9n/10.$$

Thus, **E1** holds with probability $1 - \exp(-\omega(n))$.

Now, given disjoint subsets $A, B \subset V(G)$ with size at least $n/\log^{2/5} n$, the probability there is no edge between them is

$$(1-p)^{|A||B|} = \exp(-\Omega(pn^2/\log^{4/5} n) = \exp(-\Omega(n \log^{1/5} n)).$$

Therefore, as there are at most 2^{2n} such pairs $A, B \subset V(G)$, there are no such pairs with no edges between them with probability at least $1 - 4^n \cdot \exp(-\Omega(n \log^{1/5} n)) = 1 - \exp(-\omega(n))$. That is, with probability $1 - \exp(-\omega(n))$, **E2** holds. □

Using **E1** from Proposition 3.9, we now find in a random graph a subgraph with almost the same vertex set and a good expansion property, by removing a maximal set B without this expansion property.

Lemma 3.10. *Let $c > 0$, $p = c\log n/n$, $d = \log^{1/3} n$ and $m = n/100d$. Let $V_0 \subset [n]$ be a set of at least $n/4$ vertices and $G = G(n,p)$.*

Then, with probability $1-\exp(-\omega(n))$, there is a set $B \subset [n]$ with $1 \leq |B| \leq n\log^{[3]} n/\log n$ such that, for each $U \subset V(G)\setminus B$ with $|U| \leq 2m$, we have $|N(U, V_0 \setminus B)| \geq d|U|$.

Proof. By Proposition 3.9, we have that **E1** holds in G with probability $1-\exp(-\omega(n))$. Let $B \subset V(G)$ be a largest set satisfying $|B| \leq 3m$ such that $|N(B,V_0)| \leq 2d|B|$, noting that the empty set demonstrates that such a set B exists. Note that, if $B = \emptyset$, then taking an arbitrary set $B' \subset V(G)$ with $|B'| = 1$, we have, for each $U \subset V(G)\setminus B'$ with $1 \leq |U| \leq 2m$, that $|N(U, V_0 \setminus B')| \geq 2d|U| - 1 \geq d|U|$. Therefore, we can assume that $|B| \geq 1$. We will now show that B satisfies the property in the lemma, starting with showing that, in fact, $|B| < n\log^{[3]} n/2\log n \leq m$.

As $|B| \leq 3m$ and $|N(B,V_0)| \leq 2d|B| \leq 6dm = 6n/100$, we have

$$|N(B)| \leq |V(G)\setminus V_0| + |N(B,V_0)| \leq 3n/4 + 6n/100 < 9n/10.$$

Thus, by **E1**, we have $|B| < n\log^{[3]} n/2\log n \leq m$.

Now, let $U \subset V(G)\setminus B$ with $|U| \leq 2m$ and $U \neq \emptyset$. As $|B \cup U| \leq 3m$, by the choice of B we have that $|N(B\cup U, V_0)| > 2d|B\cup U|$. Then,

$$|N(U, V_0\setminus B)| \geq |N(B\cup U, V_0)| - |N(B,V_0)| > 2d|B\cup U| - 2d|B| = 2d|U| \geq d|U|,$$

as required. □

3.6 Expansion into connection

To connect pairs of vertices efficiently with paths using expansion properties we will use the extendability techniques of Glebov, Krivelevich and Johansson [11]. These methods flexibly embed bounded-degree trees in larger graphs using certain expansion conditions, though we will only use them to find paths with specified lengths between specified vertex pairs. (More generally, see [16, Section 3.1] for a practical overview of the use of the (d,m)-extendability methods and [17] for a directed generalisation for finding consistently oriented paths.)

We first recall the key definition of (d,m)-*extendability*, using the following *inclusive neighbourhood* $N'(U)$ of a vertex set U.

Definition 3.11. For each $U \subset V(G)$, let $N'(U) = \{u \in V(G) : \exists v \in U \text{ s.t. } vu \in E(G)\} = \cup_{u\in U} N(u)$.

Definition 3.12. Let $d \geq 3$ and $m \geq 1$, let G be a graph, and let $S \subset G$ be a subgraph of G. We say that S is (d,m)-*extendable in* G if S has maximum degree at most d and, for all sets $U \subset V(G)$ with $|U| \leq 2m$,

$$|N'(U)\setminus V(S)| \geq (d-1)|U| - \sum_{x\in U\cap V(S)} (d_S(x)-1). \tag{4}$$

Given two vertices in an extendable subgraph, we can add a path with a specified length between them (subject to certain simple conditions) to, crucially, get a subgraph which is still extendable. This allows a sequence of paths to be added while remaining extendable. This is possible using the following lemma.

Lemma 3.13. *[16, Corollary 3.12] Let $d, m, \ell \in \mathbb{N}$ satisfy $m \geq 1$ and $d \geq 3$. Let $k = \lceil \log(2m)/\log(d-1)\rceil$ and $\ell \geq 2k+1$. Let G be a graph in which any two disjoint sets of size m have some edge between them. Let S be a (d,m)-extendable subgraph of G with at most $|G| - 10dm - (\ell - 2k - 1)$ vertices.*

Suppose a and b are two distinct vertices in S, both with degree at most $d/2$ in S. Then, there is an a,b-path P in G, with length ℓ and internal vertices outside of S, so that $S + P$ is (d,m)-extendable in G.

We can now prove Lemma 3.4.

Proof of Lemma 3.4. As in the statement of the lemma, let $V_0 \subset [n]$ satisfy $|V_0| \geq n/4$ and let $p = \log n/10^4 n$. Furthermore, let $d = \log^{1/3} n$ and $m = n/100d$. By Lemma 3.10 with $c = 10^{-4}$, with probability $1-\exp(-\omega(n))$, $G = G(n,p)$ contains a set $B \subset [n]$ such that the following holds.

F1 $1 \leq |B| \leq n\log^{[3]} n/\log n$ and, for any set $U \subset V(G) \setminus B$ with $|U| \leq 2m$, $|N_G(U, V_0 \setminus B)| \geq d|U|$.

By Proposition 3.9 with $c = 10^{-4}$, with probability $1 - \exp(-\omega(n))$, the following holds.

F2 Any two disjoint sets $U, U' \subset V(G)$ of size at least m have some edge between them in G.

Therefore, there is some n_0 such that, for each $n \geq n_0$, with probability at least $1-\exp(-2n)$, $G = G(n,p)$ contains a set B such that **F1** and **F2** hold, and we can assume that $d \geq 12$.

We will now show that G and B have the property in the lemma. Note that, from **F1**, we have the required bounds on $|B|$. Let then $k \geq 1$ and take any integers $\ell_i \geq 10\log n/\log^{[2]} n$, $i \in [k]$, such that $\sum_{i\in[k]} \ell_i \leq n/8$, and distinct vertices $x_1, \ldots, x_k, y_1, \ldots, y_k \in V(G) \setminus (V_0 \cup B)$. Let S_0 be the graph with vertex set $V(G) \setminus (V_0 \cup B)$ and no edges, and let $G' = G - B$. We will show that S_0 is (d,m)-extendable in G'. First note that $S_0 \subset G'$, and $\Delta(S_0) = 0$. For each $U \subset V(G')$ with $|U| \leq 2m$, we have

$$|N'_{G'}(U) \setminus V(S_0)| \geq |N_G(U, V_0 \setminus B)| \overset{\mathbf{F1}}{\geq} d|U| \geq (d-1)|U| - \sum_{x\in U\cap V(S_0)} (d_{S_0}(x) - 1),$$

as required. Thus, S_0 is (d,m)-extendable in G'.

Now, for each $i = 1, \ldots, k$ in turn, we can apply Lemma 3.13 to find a path P_i, such that

G1 P_i is an x_i, y_i-path in G' with length ℓ_i and interior vertices in $V(G') \setminus (\cup_{j=1}^{i-1} V(P_j))$, and

G2 $S_i := S_0 + P_1 + \ldots + P_i$ is (d,m)-extendable in G'.

Indeed, suppose that we seek the path P_i for some $i \in [k]$. Then, $S_{i-1} = S_0 + P_1 + \ldots + P_{i-1}$ is (d,m)-extendable in G' by **G2** for $i-1$ if $i > 1$, or as S_0 is (d,m)-extendable in G'. Now, note that

$$\begin{aligned} |S_i| &\leq |V(G') \setminus V_0| + \sum_{j=1}^{i-1} \ell_j \leq |G'| - |V_0 \setminus B| - \ell_i + \sum_{j\in[k]} \ell_j \\ &\leq |G'| - \frac{n}{4} + |B| - \ell_i + \sum_{j\in[k]} \ell_j \leq |G'| - \frac{n}{8} + |B| - \ell_i \leq |G'| - 10dm - \ell_i. \end{aligned}$$

Furthermore, $\ell_i \geq 10\log n/\log^{[2]} n \geq 2\lceil \log(2m)/\log(d-1)\rceil + 1$. Finally, note that, as $x_1, \ldots, x_k, y_1, \ldots, y_k$ are distinct, by **G1**, x_i and y_i have degree $0 \leq d/2$ in S_{i-1}. Therefore, by Lemma 3.13 and **F2**, there is an x_i, y_i-path P_i with length ℓ_i and interior vertices in $V(G') \setminus V(S_{i-1}) = V_0 \setminus (B \cup (\cup_{j<i} V(P_j)))$ and such that $S_i := S_{i-1} + P_i$ is (d,m)-extendable. Thus, **G1** and **G2** are satisfied for i.

Suppose then we have paths P_i, $i \in [k]$, satisfying **G1** and **G2**. By **G1**, for each $i \in [k]$, P_i is an x_i, y_i-path with length ℓ_i, and the paths P_i, $i \in [k]$, are internally vertex-disjoint. Furthermore, for each $i \in [k]$, the internal vertices of P_i are in $V(G') \setminus V(S_{i-1}) \subset V(G') \setminus V(S_0) = V_0 \setminus B$.

Let $V_1 = V_0 \setminus (B \cup (\cup_{j\in[k]} V(P_j))) = V_0 \setminus (B \cup V(S_k))$. To show that the paths P_i, $i \in [k]$, have the property in the lemma, it is left only to show that, for any $A \subset V(G) \setminus B$ with $V_1 \subset A$ and $A \cap B = \emptyset$, $G[A]$ is a 10-expander. Take then such a set A, and let $H := G[A] \subset G'$.

For each $U \subset V(H)$ with $0 < |U| \leq m$, by **G2** for $i = k$, as $U \subset V(H) \subset V(G')$ and $V(S_k) = (V(G) \setminus (V_0 \cup B)) \cup (\cup_{j\in[k]} V(P_j)) = V(G') \setminus V_1$, we have

$$\begin{aligned} |N_H(U)| &\geq |N'_H(U)| - |U| = |N'_{G'}(U) \cap A| - |U| \geq |N'_{G'}(U, V_1)| - |U| = |N'_{G'}(U) \setminus V(S_k)| - |U| \\ &\geq (d-1)|U| - |U| \geq (d-2)|U| \geq 10|U|. \end{aligned} \tag{5}$$

For each $U \subset V(H)$ with $m < |U| \leq |H|/20$, by **F2** we have that $|V(H) \setminus (U \cup N_H(U))| \leq m$. Therefore, using that $|U| \leq |H|/20$, we have

$$|N_H(U)| \geq |H| - |U| - m \geq |H| - 2|U| \geq 10|U|,$$

so that $|N_H(U)| \geq 10|U|$ for each $U \subset V(H)$ with $|U| \leq |H|/20$. Finally, by (5), any connected component of H must contain more than m vertices. Therefore, by **F2**, H has 1 connected component. Thus, H is connected, and hence an 10-expander, as required. □

3.7 Covering vertices with a path

We now prove Lemma 3.5. In a pseudorandom digraph D, this allows us to use sections of the cycle C to cover the exceptional set X as well as a further set B of vertices corresponding to the set B found in Lemma 3.4. To prove Lemma 3.5, we identify for some r a sequence $B_0 \subset B_1 \subset B_2 \subset \ldots \subset B_r = B$ of subsets of B, such that, roughly speaking, the later a vertex first appears in a set in the sequence, the harder it is to cover with paths in D. We then embed paths covering X while using vertices from B, before, iteratively, for each i from $r-1$ to 1, covering the unused vertices from $B \setminus B_i$ while using unused vertices in B_i if necessary. This ensures that we cover vertices which are more difficult to cover first.

Proof of Lemma 3.5. Following the lemma statement, let D be an n-vertex pseudorandom graph with exceptional set X, let $B^-, B^+, A^+, A^- \subset V(D) \setminus X$ be disjoint, such that X, B^+ and B^- each have size at most $n \log^{[3]} n / \log n$, and such that, for each $v \in V(D)$ and $\diamond \in \{+,-\}$, we have $d^\diamond(v, B^\diamond \cup A^\diamond) \geq \log n / 5000$. Let $B = B^+ \cup B^-$ and let P_i, $i \in [k]$, be vertex-disjoint oriented paths with length $2\lceil 4 \log n / \log\log n \rceil$, where $k = |X \cup B|$ and, for each $i \in [k]$, x_i is the midpoint of P_i. Let $f : X \cup B \to [k]$ be a bijection.

Let $d = \log n / 10^4$ and let $B_0^+ = B_0^- = \emptyset$. Iteratively, for each integer $i \geq 1$ and $\diamond \in \{+,-\}$, let

$$B_i^\diamond = \{v \in B^\diamond : d^+(v, B_{i-1}^+ \cup A^+) \geq d \text{ and } d^-(v, B_{i-1}^- \cup A^-) \geq d\}. \tag{6}$$

This gives a sequence $B_0^+ \subset B_1^+ \subset \ldots$ of subsets of B^+ and a sequence $B_0^- \subset B_1^- \subset \ldots$ of subsets of B^-. For each $i \geq 0$, let $B_i = B_i^+ \cup B_i^-$. We will show the following claim.

Claim 3. For each $i \geq 0$, $|B \setminus B_i| \leq |B| / (\log n)^{i/3}$.

Proof. Note that this is true for $i = 0$. We will prove this by induction on i, so suppose that $i > 0$ and that it is true for $i - 1$. For each $\diamond \in \{+,-\}$, let $Z_i^\diamond$ be the set of vertices $v \in B$ such that $d^\diamond(v, B_{i-1}^\diamond \cup A^\diamond) < d$. Observe that, by (6), $Z_i^+ \cup Z_i^- = B \setminus B_i$. Furthermore, for each $\diamond \in \{+,-\}$, every vertex $v \in Z_i^\diamond$ has $d^\diamond(B^\diamond \cup A^\diamond) \geq \log n / 5000 = 2d$ but $d^\diamond(B_{i-1}^\diamond \cup A^\diamond) < d$. Therefore, for each $v \in Z_i^\diamond$, we have $d^\diamond(v, B^\diamond \setminus B_{i-1}^\diamond) > d \geq (\log n)^{2/3}$. Therefore, by **A3** in the definition of pseudorandomness, as $|Z_i^\diamond| \leq |B| \leq 2n \log^{[3]} n / \log n$, we have $|B^\diamond \setminus B_{i-1}^\diamond| \geq |Z_i^\diamond| (\log n)^{1/3}$. Therefore,

$$\begin{aligned} |B \setminus B_i| = |Z_i^+ \cup Z_i^-| &\leq |Z_i^+| + |Z_i^-| \leq (|B^+ \setminus B_{i-1}^+| + |B^- \setminus B_{i-1}^-|) / (\log n)^{1/3} \\ &= |B \setminus B_{i-1}| / (\log n)^{1/3} \leq |B| / (\log n)^{i/3}. \end{aligned}$$

This completes the inductive step, and hence the proof of the claim. □

By Claim 3, if $i \geq 3(\log n / \log\log n) + 1$, then $B \setminus B_i = \emptyset$. Let then $r \leq 4 \log n / \log\log n$ be the smallest integer such that $B \setminus B_r = \emptyset$, and note that $B_r^+ = B^+$ and $B_r^- = B^-$. For each $\diamond \in \{+,-\}$, let $H^\diamond$ be the bipartite auxiliary (undirected) graph with vertex classes a copy of $X \cup B$ and a disjoint copy of $B^\diamond \cup A^\diamond$, with an edge xy between $x \in X \cup B$ and $y \in B^\diamond \cup A^\diamond$ if, for some $i \in \{0, 1, \ldots, r\}$, $x \in X \cup (B \setminus B_i^\diamond)$ and $y \in B_i^\diamond \cup A^\diamond$, and $y \in N_D^\diamond(x)$.

Claim 4. For each $\diamond \in \{+,-\}$ and $U \subset X \cup B$, we have $|N_{H^\diamond}(U)| \geq 2|U|$.

Proof of Claim 4. Let $\diamond \in \{+,-\}$ and $U \subset X \cup B$. Suppose $x \in U$. If $x \in U \cap X$, then, as $B_r^\diamond = B^\diamond$, for each $y \in (B^\diamond \cup A^\diamond) \cap N_D^\diamond(x)$, $xy \in E(H^\diamond)$. Thus, we have $d_{H^\diamond}(x) \geq \log n / 5000 > d$.

On the other hand, if $x \in U \cap B$, then let i be the smallest $i \in [r]$ with $x \in B_i^+ \cup B_i^-$. Then, by the choice of B_i^+ and B_i^-, we have $d_D^\diamond(x, B_{i-1}^\diamond \cup A^\diamond) \geq d$. Thus, we have $d_{H^\diamond}(x) \geq d$.

Therefore, $d_{H^\diamond}(x) \geq d$ for each $x \in U$. Let $V = N_{H^\diamond}(U)$, and let U' and V' be the vertex sets in D which correspond to U and V. We have that $d_D^\diamond(x, V') \geq d$ for each $x \in U'$, and therefore, by **A3**, as $|U| \leq |X \cup B| \leq 3n \log^{[3]} n / \log n$, we have that

$$|N_{H^\diamond}(U)| = |V| = |V'| \geq |U'| (\log n)^{1/3} \geq 2|U'| = 2|U|,$$

as required. □

Therefore, for each $\diamond \in \{+,-\}$, the appropriate Hall's matching criterion holds by Claim 4 to show that there exist functions $g_1^\diamond, g_2^\diamond : X \cup B \to B^\diamond \cup A^\diamond$ so that $vg_i(v) \in E(H^\diamond)$ for each $v \in X \cup B$ and $i \in [2]$, and $g_1^\diamond(v), g_2^\diamond(v)$, $v \in X \cup B$, are all distinct vertices in $B^\diamond \cup A^\diamond$.

For each $v \in X \cup B$ we now find a path Q_v covering v, in which $f(x)$ is copied to x and then, moving in either direction on Q_v, the vertices appear earlier and earlier in the sequence $A^+ \cup A^- \cup B_0, B_1, \ldots, B_r$. To do this, for each $v \in X \cup B$, let $Q_v \subset D$ be a longest path satisfying the following properties.

H1 Q_v is a copy of a portion of $P_{f(v)}$ containing $x_{f(v)}$ in which $x_{f(v)}$ is copied to v.

H2 Each interior vertex of Q_v is in $X \cup B$, and the endvertices of Q_v are in $X \cup B \cup A^+ \cup A^-$.

H3 For each $u \in V(Q_v)$ and $\diamond \in \{+,-\}$, if w is a $\diamond$-neighbour of u in Q_v which lies further from v than u on the underlying undirected path of Q_v (if such a w exists), then $w \in \{g_1^\diamond(u), g_2^\diamond(u)\}$.

Note that the path Q_v consisting solely of the vertex v satisfies these conditions, so such a path Q_v does exist.

We now pick a subcollection of these paths which are disjoint. To do this, iteratively, for each $i = r, \ldots, 1$, let

$$\bar{B}_i = \{v \in B_i \setminus B_{i-1} : v \notin V(Q_u) \text{ for each } u \in X \cup \bar{B}_r \cup \bar{B}_{r-1} \cup \ldots \cup \bar{B}_{i+1}\}.$$

Let $\bar{B} = \bar{B}_r \cup \ldots \cup \bar{B}_1$. We will show that the paths Q_v, $v \in X \cup \bar{B}$, satisfy the conditions in the lemma. That is, that they are vertex-disjoint and satisfy **B1**–**B2**.

Note first that, by the choice of $\bar{B}$, the paths Q_v, $v \in X \cup \bar{B}$, contain every vertex in $X \cup B$, and thus **B2** holds. As **H1** holds, to show that **B1** holds it is sufficient to show that, for each $v \in X \cup B$, the endvertices of Q_v are in $A^+ \cup A^-$. Therefore, to complete the proof of the lemma we need only show the following two claims.

Claim 5. For each $v \in X \cup B$, the endvertices of Q_v are in $A^+ \cup A^-$.

Claim 6. The paths Q_v, $v \in X \cup \bar{B}$, are vertex-disjoint.

Before proving these claims, we will deduce two properties, **I1** and **I2** that we require. For each vertex $v \in X \cup B$, let i_v be the largest integer $i \in \{0, 1, \ldots, r\}$ such that $v \in X \cup (B \setminus B_i)$. Note that, if $v \in X \cup B$, $\diamond \in \{+,-\}$, and $x, y \in V(Q_v)$, and x is closer to v on Q_v than y (and possibly even $x = v$) and y is a $\diamond$-neighbour of x on Q_v, then, by **H3**, $y \in \{g_1^\diamond(x), g_2^\diamond(x)\}$, so that $xy \in E(H^\diamond)$. Thus, by the definition of $H^\diamond$, if, in addition, $x, y \in X \cup B$, then $i_y < i_x$. Thus, we have the following two properties.

I1 If $v \in X \cup B$, and $x, y \in (X \cup B) \cap V(Q_v)$, are such that x is closer to v on Q_v than y, then $i_y < i_x$.

I2 If $v \in X \cup B$, and $x \in (X \cup B) \cap V(Q_v)$, then any vertex on Q_v in $\{g_1^+(x), g_2^+(x), g_1^-(x), g_2^-(x)\}$ is a neighbour of x on Q_v which is further than v from x on Q_v, and every such neighbour must be in $\{g_1^+(x), g_2^+(x), g_1^-(x), g_2^-(x)\}$.

We now prove Claim 5 and 6.

Proof of Claim 5. Suppose, to the contrary, that there is some $v \in X \cup B$ such that Q_v has some endvertex, w say, which is not in $A^+ \cup A^-$. Let $P'_{f(v)} \subset P_{f(v)}$ be the subpath of $P_{f(v)}$ of which Q_v is a copy, and let its endvertex of which w is a copy be w'. Note that, by **I1**, there are at most $r-1$ vertices between v and w on Q_v. Thus, we can pick $x' \in V(P_{f(v)}) \setminus V(P'_{f(v)})$ and $\diamond \in \{+,-\}$ be such that x' is a $\diamond$-neighbour of w on $P_{f(v)}$. By **I2**, Q_v contains at most one vertex in $\{g_1^+(w), g_2^+(w), g_1^-(w), g_2^-(w)\} \subset B \cup A^+ \cup A^-$ (and no such vertex if $w \neq v$). Therefore, we can pick $x \in \{g_1^\diamond(w), g_2^\diamond(w)\} \setminus V(Q_v)$. Noting that $Q'_v = Q_v + wx$ satisfies **H1**–**H3** in place of Q_v contradicts the maximality of Q_v. □

Proof of Claim 6. Suppose, to the contrary, that there are distinct $u, v \in X \cup \bar{B}$, and a vertex $x \in V(Q_u) \cap V(Q_v)$. Assume further that x is as close to u as possible on Q_u subject to $x \in V(Q_u) \cap V(Q_v)$.

Suppose first that $u \neq x$ and $v \neq x$. By **I2**, there is some $x_u \in V(Q_u)$ which is closer to x on Q_u than x is and for which $x \in \{g_1^+(x_u), g_2^+(x_u), g_1^-(x_u), g_2^-(x_u)\}$. Similarly, there is some $x_v \in V(Q_v)$ which is closer to v on Q_v than x is and for which $x \in \{g_1^+(x_v), g_2^+(x_v), g_1^-(x_v), g_2^-(x_v)\}$. As the sets

$\{g_1^+(w), g_2^+(w), g_1^-(w), g_2^-(w)\}$, $w \in X \cup B$, are disjoint, we have $x_v = x_u$, contradicting the assumption on x.

Therefore, by swapping the labels of u and v if necessary, we can assume, as u and v are distinct that $u = x$ and $v \neq x$. By the choice of $\bar{B}_{i_u}$, as $u, v \in \bar{B}$ and $u \in V(Q_v)$, we must have that $i_v \leq i_u$. However, by **I1**, as v is closer to v on Q_v than $u \in V(Q_v) \setminus \{v\}$ is, we have $i_u < i_v$, a contradiction. This completes the proof of the claim, and hence the lemma. □ □

3.8 Pósa rotation and extension

We will now prove Lemma 3.6, using a standard implementation of Pósa's rotation-extension technique with edge sprinkling to find a Hamilton cycle (see, for example, [5]). We include the proof to record the very high probability of success that we need, as well as to show that a Hamilton cycle can be found including any fixed edge e, to then have a Hamilton path between the vertices of e. We begin by defining an e-booster, and a rotation.

Definition 3.14. For a graph G and edges $e, f \in V(G)^{(2)}$, f is an e-booster for G if either $G + e + f$ has a longer path containing e than $G + e$ does, or $G + e + f$ contains a Hamilton cycle through e.

Definition 3.15. Let a path Q in a graph G be $Q = u_0 u_1 \dots u_\ell$. Let $e \in E(Q)$ and $0 \leq i \leq \ell - 1$ with $u_i u_{i+1} \neq e$. Then, we *rotate Q in G with u_0 and e fixed using $u_\ell u_i$* to get the path $(Q - u_i u_{i+1}) + u_\ell u_i$.

Note that this is a u_0, u_{i+1}-path containing e with the same vertex set as Q.

We now show that a 10-expander has many boosters.

Lemma 3.16. *Let $n \geq 3$. If an n-vertex graph G is a 10-expander, and $e \in V(G)^{(2)}$, then G has at least $n^2/10^4$ e-boosters.*

Proof. Note that, by Definition 3.14, we can assume that $e \in E(G)$. Let P be a maximal path in G containing e, and $V = V(P)$. Let E be the set of pairs ab for which there is an a, b-path, P_{ab} say, in G containing e with vertex set V. Note that each $ab \in E$ is an e-booster for G. Indeed, $P_{ab} + ab$ is a cycle in G with vertex set V which contains e. If $V = V(G)$, then $P_{ab} + ab$ is a Hamilton cycle containing e. If $V \neq V(G)$, then, as G is a 10-expander, and hence connected, there exists some $x \in V(G) \setminus V$ and $y \in V$ with $xy \in E(G)$. Let e' be an edge of $P_{ab} + ab$ containing y which is not e. Then, $P_{ab} + ab + xy - e'$ is a path containing e with length greater than P in $G + ab$. Thus, in both cases, ab is an e-booster for G.

Suppose then, for contradiction, that $|E| \leq n^2/10^4$. As the pair of endvertices of P is in E, $E \neq \emptyset$, so there is some vertex u in a pair in E. If u is in more than $n/50$ such pairs, then there are at least $n/50$ vertices in some pair in E. Therefore, by averaging if necessary, there is some $u_0 \in V$ such that $1 \leq |\{v \in V : u_0 v \in E\}| \leq n/50$. Let Q be a path in G with vertex set V which contains e and has u_0 as an endvertex. Let $V_0 \subset V$ be the set of vertices $v \in V \setminus \{u_0\}$ such that a u_0, v-path with vertex set V can be reached by iteratively rotating Q in G with u_0 and e fixed. Note that $|V_0| \leq |\{v : u_0 v \in E\}| \leq n/50$.

For each $v \in V_0$, let Q_v be a u_0, v-path with vertex set V which can be reached by iteratively rotating Q with u_0 and e fixed. For each $v \in V_0$, as Q_v is a maximal path in G, we have $N(v) \subset V(Q_v) = V$, and thus $N(V_0) \subset V$.

Let ℓ be the length of Q and label Q as $u_0 u_1 \cdots u_\ell$. Note that, for each $i \in [\ell - 1]$, if $u_{i-1}, u_i, u_{i+1} \notin V_0$ then in any sequence of rotations fixing e and u_0 we always preserve the subpath $u_{i-1} u_i u_{i+1}$ (in either order), and therefore we never rotate using an edge containing u_i. Thus, for each $i \in [\ell - 1]$, if $v \in V_0$ and $u_i v \in E(G)$, then we must have one of $u_{i-1} \in V_0$, $u_i \in V_0$, $u_{i+1} \in V_0$, $u_{i-1} u_i = e$ or $u_i u_{i+1} = e$. There can be at most $3|V_0| + 2$ values of i for each v, and hence $|N_G(V_0)| \leq 3|V_0| + 2 < 10|V_0|$. As G is a 10-expander, $|V_0| > n/20$, contradicting that $|V_0| \leq n/50$. □

We use the following standard form of Azuma's inequality for a sub-martingale (see, for example [2], for an exposition of martingales and Azuma's inequality).

Theorem 3.17 (Azuma's inequality). *If $X_0, X_1, \dots, X_n$ is a sub-martingale, and $|X_i - X_{i-1}| \leq c_i$ for each $1 \leq i \leq n$, then, for each $t > 0$, $\mathbb{P}(X_n - X_0 \leq -t) \leq \exp\left(\frac{-t^2}{2\sum_{i=1}^n c_i^2}\right)$.*

We now prove Lemma 3.6.

Proof of Lemma 3.6. As in the lemma statement, let G_0 be a 10-expander with vertex set $[n]$ and let $x, y \in V(G_0)$ be distinct. Let $p = \log n/10^5 n$ and $G_1 = G(n,p)$. We will show that, with probability $1-\exp(-\omega(n))$, $G_0 \cup G_1$ contains a Hamilton x, y-path, and thus the lemma follows.

Let $m = E(G_1)$ and, uniformly at random, label $E(G_1) = \{e_1, \ldots, e_m\}$. Let $m_0 = \log n/10^6$. By a simple application of Lemma 3.8, we have

$$\mathbb{P}(m \geq m_0) = 1 - \exp(-\omega(n)). \tag{7}$$

We will show that, letting E be the event that $G_0 \cup G_1$ contains an x, y-Hamilton cycle, we have

$$\mathbb{P}(E|m = m_0) = 1 - \exp(-\omega(n)). \tag{8}$$

As $\mathbb{P}(E|m = \bar{m}) \geq \mathbb{P}(E|m = m_0)$ for each $\bar{m} \geq m_0$, we will then have

$$\begin{aligned} \mathbb{P}(E) = \sum_{\bar{m}=0}^{\binom{n}{2}} \mathbb{P}(m = \bar{m}) \cdot \mathbb{P}(E|m = \bar{m}) &\geq \sum_{\bar{m}=m_0}^{\binom{n}{2}} \mathbb{P}(m = \bar{m}) \cdot \mathbb{P}(E|m = m_0) \\ &\geq \mathbb{P}(m \geq m_0) \cdot \mathbb{P}(E|m = m_0) \overset{(7),(8)}{=} 1 - \exp(-\omega(n)), \end{aligned}$$

as required.

It is left then to prove (8). Let $H_0 = G_0$, and, for each $1 \leq i \leq m_0$, let $H_i = H_{i-1} + e_i$ and let X_i be 1 if e_i is an xy-booster for H_{i-1}, and 0 otherwise. For each $i \in [m_0]$, as H_{i-1} contains H_0 it is always a 10-expander. Hence, by Lemma 3.16, the set of xy-boosters for H_{i-1} always has size at least $n^2/10^4$. Therefore, the probability that $X_i = 1$, conditioned on any possible values of $H_1, \ldots, H_{i-1}$ is at least $(n^2/10^4 - i)/\binom{n}{2} \geq 1/10^5$. For each $0 \leq i \leq m_0$, let $Y_i = \sum_{j=1}^{i}(X_j - 1/10^5)$, and note that $Y_0, Y_1, \ldots, Y_{m_0}$ is a sub-martingale with $|Y_i - Y_{i-1}| \leq 1$ for each $i \in [m_0]$

Thus, by Theorem 3.17,

$$\mathbb{P}(Y_{m_0} < -m_0/10^6) = \exp(-\Omega(m_0)) = \exp(-\omega(n)).$$

Note that, if $Y_{m_0} \geq -m_0/10^6$, then $\sum_{i=0}^{m_0} X_i \geq 9m_0/10^6 = \omega(n)$. Furthermore, if $\sum_{i=0}^{m_0} X_i \geq n$, then at least n xy-boosters are added somewhere in the sequence $H_1, \ldots, H_{m_0}$, and hence H_{m_0} contains a Hamilton cycle containing xy, and thus a Hamilton x, y-path. Therefore, (8) holds, and the proof is complete. □

4 Proof of Theorem 2.4

We will prove Theorem 2.4 from Theorem 2.3 in Section 4.3, and then deduce Theorems 1.1 and 1.3 in Sections 4.4 and 4.5. We start in Section 4.1 by proving some properties of the random digraph process that we require, before selecting the vertices from the cycle to be embedded in Section 4.2.

4.1 Properties of the random digraph process

In the n-vertex random digraph process $D_0, D_1, \ldots, D_{n(n-1)}$, we focus on four particular digraphs, D_{i_0}, D_{i_1}, D_{i_2} and D_{i_3}, where

$$i_0 = \frac{9n \log n}{20}, \quad i_1 = \frac{n \log n}{2} - n \log \log n, \quad i_2 = \frac{3n \log n}{4} \quad \text{and} \quad i_3 = n \log n + 2n \log \log n. \tag{9}$$

We will prove that the properties **J1**–**J12** typically hold for these digraphs, in Lemmas 4.2 and 4.3, where, for convenience, we use the random digraphs K_0, K_1, K_2 and K_3. The digraphs we need to consider for Theorem 2.4 will lie between D_{i_1} and D_{i_3}. That is, D_{i_1} has, with high probability, some vertex v with $d^+(v) + d^-(v) \leq 1$ (see **J10**), and D_{i_3} has, with high probability, minimum in- and out-degree at least 2 (see **J12**). We use D_{i_0} as a point in the random digraph process by which most vertices do not have low in- or out-degree, and D_{i_2} as a point by which the low degree vertices are likely to be few enough that they are well-spaced in the digraph. In what follows, we consider 'low degree' to be degree at most $\log n/300$.

After adding i_0 edges in the random digraph process, giving the digraph D_{i_0}, it will typically already be clear that most of the vertices in each D_i, $i_1 \le i \le i_3$, will not have low in- or out-degree. We collect the vertices for which this is not yet guaranteed into the set S_0 (see (10)), which is likely to have size at most $n^{2/3}$ (see **J1**). To determine the vertices of low in- and out-degree of each D_i, $i_1 \le i \le i_3$, we only need to consider the edges in $E(D_i) \setminus E(D_{i_0})$ with at least one vertex in S_0 – say the set of these edges is E_i. Conditioning on $E(D_{i_0})$ and E_i, let $m_i = |E(D_{i_0}) \cup E_i|$. Observe that, with this conditioning, $E(D_i) \setminus (E(D_{i_0}) \cup E_i)$ is distributed as $i - m_i$ edges chosen uniformly at random from the non-edges in $E(D_{i_0})$ with no vertex in S_0. As S_0 is a sublinear set of vertices, when $i \ge i_1$, we are likely to have that $i - m_i = \Omega(\log n)$ (see **J2**). These $i - m_i$ edges will provide the random digraph which we use (in a modified form) to apply Theorem 2.3. The pseudorandom digraph in this application will (suitably modified) be the digraph D_i with the addition of any edge from $E(D_i) \setminus E(D_{i_0})$ with at least one vertex in S_0. As a technique, conditioning on (events including) S_0 in this way comes from work of Krivelevich, Lubetzky and Sudakov [14] in their study of the Hamiltonicity of the k-core.

We use i_2 as an arbitrary midpoint in the interval $[i_1, i_3]$ by which the relevant structure of the random digraph will have changed. In D_{i_1}, we expect to have plenty of vertices with out-degree 0 or with in-degree 0, and some vertices with out- and in-degree both 1. Our methods are complicated by the likely existence of edges (and short paths) between the vertices with low in- or out-degree (we give a likely upper-bound for such paths in **J3**). We deal with this by showing that, for $i_1 \le i \le i_2$, D_i will likely have sufficiently many vertices with in-degree 0 (see **J11**) that any cycle we embed into D_i has enough changes of direction to cover not only the vertices of in-degree 0 or out-degree 0, but also any vertices in edges and short paths between vertices with low in- or out-degree. Helpfully, there are typically never any edges or short paths between vertices with both low in- and out-degree (see **J6**), or any short cycles containing a vertex of low in- or low out-degree (see **J4**).

When enough edges are added to reach D_{i_2}, the set of low in- or out-degree vertices will have decreased in size enough that there are likely to be no edges or very short paths between these vertices, even after more edges are added to reach D_{i_3} (see **J5**). This means that low in- and out-degree vertices are sufficiently far apart in D_i, $i_2 \le i \le i_3$, that we can assign them neighbours without worrying about conflicts.

In addition to the properties mentioned above, we prove properties **J7**–**J9** are likely to hold, which we will use to help show pseudorandom properties of a modified subgraph of D_i, for each $i_1 \le i \le i_3$. We start with the following useful proposition.

Proposition 4.1. *There is some n_0 such that the following holds for each $n \ge n_0$. Letting $d = \log n/300$, for each $k \le n^{1/2}$ and p with $\log n/4n \le p \le 2\log n/n$, we have*

$$\sum_{i=0}^{d} \binom{n}{i} p^i (1-p)^{n-k} \le \exp\left(-pn + \frac{1}{30}\log n\right).$$

Proof. As $100 < enp/d \le 600e$, we have

$$\begin{aligned}\sum_{i=0}^{d} \binom{n}{i} p^i (1-p)^{n-k} &\le \sum_{i=0}^{d} \left(\frac{enp}{i}\right)^i (1-p)^{n-k} \le (d+1)\left(\frac{enp}{d}\right)^d e^{-p(n-k)} \\ &\le (700e)^d e^{-p(n-k)} \le \exp(-pn + pk + d\log(700e)) \le \exp(-pn + (\log n)/30),\end{aligned}$$

where the last line of inequalities hold for sufficiently large n as $pk \le 2n^{-1/2}\log n$ and $\log(700e) < 8$. □

We now prove that properties **J1**–**J9** are likely to hold, as follows.

Lemma 4.2. *Let (K_0, K_1, K_2, K_3) be drawn uniformly at random from the set of such tuples such that, for each $j \in \{0,1,2,3\}$, K_j is a digraph with vertex set $[n]$ and i_j edges (as set in (9)), and $K_0 \subset K_1 \subset K_2 \subset K_3$. Let $d = \log n/300$, and, for each $j \in \{0,1,2,3\}$, let*

$$S_j = \{v \in [n] : d^+_{K_j}(v) \le d \;\text{ or }\; d^-_{K_j}(v) \le d\}. \tag{10}$$

Let

$$T = \{v \in [n] : d^+_{K_1}(v) \le d \ \text{ and } \ d^-_{K_1}(v) \le d\}.$$

Then, with high probability, the following hold.

J1 $|S_0| \le n^{2/3}$.

J2 $E(K_3) \setminus E(K_0)$ *contains at most* n *edges with some vertex in* S_0.

J3 *The number of paths in* K_3 *between vertices in* S_1 *with length at most 4 is at most* $n^{1/6}$.

J4 *There are no cycles in* K_3 *with length at most 3 containing a vertex in* S_1.

J5 $K_3[S_2 \cup N^+_{K_3}(S_2) \cup N^-_{K_3}(S_2)]$ *is the disjoint union of* $|S_2|$ *stars.*

J6 $K_3[T \cup N^+_{K_3}(T) \cup N^-_{K_3}(T)]$ *is the disjoint union of* $|T|$ *stars with no vertices in* $S_1 \setminus T$.

J7 *Each* $v \in [n]$ *has at most 2 in- or out-neighbours in* K_3 *in* $S_1 \cup N^+_{K_3}(S_1 - v) \cup N^-_{K_3}(S_1 - v)$.

J8 *For any sets* $A, B \subset [n]$ *and* $\diamond \in \{+, -\}$ *with* $|A| \le 100n \log\log n/\log n$ *and, for each* $v \in A$, $d^\diamond_{K_3}(v, B) \ge (\log n)^{2/3}/2$, *we have* $|B| \ge 100|A|(\log n)^{1/3}$.

J9 $\Delta^\pm(K_3) \le 50 \log n$.

Proof. First, we will choose binomial random digraphs $\bar{K}_0, \bar{K}_1, \bar{K}_2$, and $\bar{K}_3$ and use them to choose the digraphs K_0, K_1, K_2, and K_3 with the distribution in the lemma. This will allow us to prove likely properties of $\bar{K}_0, \bar{K}_1, \bar{K}_2$, and $\bar{K}_3$ and infer **J1**–**J9** from them.

Let $N = n(n-1)$, $p_0 = (i_0 - n)/N$, $p_1 = (i_1 - n)/N$, $p_2 = (i_2 - n)/N$, and $p_3 = (i_3 + n)/N$. For each $u, v \in [n]$ with $u \ne v$, let X_{uv} be chosen uniformly at random from $[0, 1]$. For each $j \in \{0, 1, 2, 3\}$, let $\bar{K}_j$ be the digraph with vertex set $[n]$ and edge set $\{uv : u, v \in [n], u \ne v, X_{uv} \le p_j\}$. Let $d = \log n/300$. For each $j \in \{0, 1, 2, 3\}$, let $\bar{S}_j = \{v \in [n] : d^+_{\bar{K}_j}(v) \le d \text{ or } d^-_{\bar{K}_j}(v) \le d\}$. Let $\bar{T} = \{v \in [n] : d^+_{\bar{K}_1}(v) \le d \text{ and } d^-_{\bar{K}_1}(v) \le d\}$.

Now, note that, for each $j \in \{0, 1, 2, 3\}$, $\bar{K}_j$ has the same distribution as $D(n, p_j)$. Furthermore, $\bar{K}_0 \subset \bar{K}_1 \subset \bar{K}_2 \subset \bar{K}_3$. Let $\ell = e(\bar{K}_3) - e(\bar{K}_0)$, and label the edges of $E(\bar{K}_3) \setminus E(\bar{K}_0)$ as $e_1, \dots, e_\ell$ uniformly at random subject to the restriction that the edges of $E(\bar{K}_1) \setminus E(\bar{K}_0)$ come first in this order, followed by those in $E(\bar{K}_2) \setminus E(\bar{K}_1)$, and then those in $E(\bar{K}_3) \setminus E(\bar{K}_2)$. Let **E** be the event that $e(\bar{K}_3) \ge i_3$, and, for each $j \in \{0, 1, 2\}$, $e(\bar{K}_j) \le i_j$. If **E** holds, then, for each $j \in \{0, 1, 2, 3\}$, let K_j be the random graph $\bar{K}_0$ with the edges $e_1, \dots, e_{i_j - e(\bar{K}_0)}$ added. If **E** does not hold, then let (K_0, K_1, K_2, K_3) be drawn uniformly at random from the set of such tuples such that, for each $j \in \{0, 1, 2, 3\}$, K_j is a digraph with vertex set $[n]$ and i_j edges, and $K_0 \subset K_1 \subset K_2 \subset K_3$. Note that the distribution of (K_0, K_1, K_2, K_3) is the same conditioned on **E** holding or on **E** not holding, and thus has the distribution as described in the lemma.

Let $\overline{\mathbf{J1}}$–$\overline{\mathbf{J9}}$ be the properties **J1**–**J9** with K_j, S_j and T replaced by $\bar{K}_j$, $\bar{S}_j$ and $\bar{T}$ respectively, for any relevant $j \in \{0, 1, 2, 3\}$. Now, if **E** holds, then $\bar{K}_0 \subset K_0$, $\bar{K}_1 \subset K_1$, $\bar{K}_2 \subset K_2$, and $K_3 \subset \bar{K}_3$ and we also have $T \subset \bar{T}$, and $S_j \subset \bar{S}_j$ for each $j \in \{0, 1, 2\}$. Therefore, if **E** holds, then each property $\overline{\mathbf{J1}}$–$\overline{\mathbf{J9}}$ implies the corresponding property **J1**–**J9**. Indeed, decreasing the sets T, S_0, S_2, adding edges in K_3 to K_0 and removing edges from K_3 makes it easier for each of these properties to hold. Therefore, if **E** and $\overline{\mathbf{J1}}$–$\overline{\mathbf{J9}}$ hold individually with high probability, then **J1**–**J9** hold collectively with high probability. We now show in turn that each of **E** and $\overline{\mathbf{J1}}$–$\overline{\mathbf{J9}}$ hold with high probability.

E: By Lemma 3.8 with $\varepsilon = 1/4 \log n$, for each $j \in \{0, 1, 2\}$,

$$\begin{aligned} \mathbb{P}(e(\bar{K}_j) > i_j) &\le \mathbb{P}(|e(\bar{K}_j) - p_j n(n-1)| > n) \le \mathbb{P}(|e(\bar{K}_j) - p_j n(n-1)| \ge \varepsilon p_j n(n-1)) \\ &\le 2\exp(-\varepsilon^2 p_j n(n-1)/3) = o(1). \end{aligned} \tag{11}$$

Similarly, $\mathbb{P}(e(\bar{K}_3) < i_3) = o(1)$. Therefore, **E** holds with high probability.

$\overline{\mathbf{J1}}$**:** For each $v \in \bar{S}_0$, there will be some $\diamond \in \{+,-\}$ and $A \subset [n] \setminus \{v\}$ with $|A| \le d$ such that there is a $\diamond$-edge from v to each vertex in A and no $\diamond$-edge from A to $[n] \setminus (A \cup \{v\})$. Therefore, for large n, using Proposition 4.1, we have, as $p_0 n = (9/20 - o(1)) \log n$, that

$$\mathbb{E}|\bar{S}_0| \le n \cdot 2 \cdot \sum_{i=0}^{d} \binom{n-1}{i} p_0^i (1-p_0)^{n-i-1} \le 2n \cdot \exp(-p_0 n + (\log n)/30) = o(n^{2/3}).$$

Therefore, by Markov's inequality, with high probability $\overline{\mathbf{J1}}$ holds.

$\overline{\mathbf{J2}}$**:** We will show that $\mathbb{P}(\overline{\mathbf{J2}} \text{ holds} \,|\, \overline{\mathbf{J1}} \text{ holds}) = 1 - o(1)$. Revealing the edges of $\bar{K}_0$, if $\overline{\mathbf{J1}}$ holds, then there are at most $2|\bar{S}_0| \cdot n \le 2n^{5/3}$ directed non-edges in $\bar{K}_0$ with at least one vertex in $\bar{S}_0$. For each such uv, the probability that $uv \in E(\bar{K}_3) \setminus E(\bar{K}_0)$ is $\mathbb{P}(X_{uv} \le p_3 | X_{uv} > p_0) \le 2p_3$. Thus, the number of edges in $E(\bar{K}_3) \setminus E(\bar{K}_0)$ has expectation at most $4p_3 n^{5/3} = o(n)$. Therefore, $\mathbb{P}(\overline{\mathbf{J2}} \text{ holds} \,|\, \overline{\mathbf{J1}} \text{ holds}) = 1 - o(1)$. Thus, as $\overline{\mathbf{J1}}$ holds with high probability, so does $\overline{\mathbf{J2}}$.

$\overline{\mathbf{J3}}$**:** Let X_1 be the number of paths of length at most 4 in $\bar{K}_3$ between vertices in $\bar{S}_1$. For each such path, there are distinct vertices x, y (for the endvertices) an integer $k \in \{0,1,2,3\}$, and distinct vertices $v_1, \ldots, v_k \in [n] \setminus \{x,y\}$, so that $xv_1 \ldots v_k y$ is a path in $\bar{K}_3$ (with any orientations on its edges), and (as $x, y \in \bar{S}_1$) sets $A_x, A_y \subset [n] \setminus \{x, y, v_1, \ldots, v_k\}$ with size at most d and $\diamond_x, \diamond_y \in \{+,-\}$ for which, for each $v \in \{x,y\}$ there is an $\diamond_v$-edge from v to each vertex in A_v in $\bar{K}_1$ and no $\diamond_v$-edge from v to $[n] \setminus (A_v \cup \{x, y, v_1, \ldots, v_k\})$ in $\bar{K}_1$. Thus, as there are 2^{k+1} possible orientations for a path with k interior vertices, for large n, using Proposition 4.1, we have, as $2p_1 n = (1 - o(1)) \log n$, that

$$\begin{aligned}
\mathbb{E}X_1 &\le \frac{n(n-1)}{2} \cdot \sum_{k=0}^{3} \binom{n-2}{k} 2^{k+1} p_3^{k+1} \cdot \left(2 \sum_{i=0}^{d} \binom{n-2-k}{i} p_1^i (1-p_1)^{n-2-k-i} \right)^2 \\
&\le n \cdot \sum_{k=0}^{3} (2p_3 n)^{k+1} \cdot 4 \exp(-2p_1 n + (\log n)/15) \\
&\le 16n \cdot (4 \log n)^4 \cdot \exp(-(9/10 + o(1)) \log n) = o(n^{1/6}). \qquad (12)
\end{aligned}$$

Thus, with high probability $\overline{\mathbf{J3}}$ holds.

$\overline{\mathbf{J4}}$**:** Let X_2 be the number of cycles of length at most 3 in $\bar{K}_3$ with a vertex in $\bar{S}_1$. Thus, as there are 2^{k+1} possible orientations for a cycle with $k+1$ vertices, for large n, using Proposition 4.1, we have

$$\begin{aligned}
\mathbb{E}X_2 &\le n \cdot \sum_{k=1}^{2} \binom{n-1}{k} 2^{k+1} p_3^{k+1} \cdot \left(2 \sum_{i=0}^{d} \binom{n-1-k}{i} p_1^i (1-p_1)^{n-1-k-i} \right) \\
&\le \sum_{k=1}^{2} (2p_3 n)^{k+1} \cdot 2 \exp(-p_1 n + (\log n)/30) \le 4 \cdot (4 \log n)^3 \cdot \exp(-(\log n)/4) = o(1), \qquad (13)
\end{aligned}$$

where we have used that $p_1 n = (1/2 - o(1)) \log n$. Therefore, with high probability $\overline{\mathbf{J4}}$ holds.

$\overline{\mathbf{J5}}$**:** Let X_3 be the number of paths of length at most 4 in $\bar{K}_3$ between vertices in $\bar{S}_2$. By a similar calculation to (12), we have $\mathbb{E}X_3 \le 16n \cdot (4 \log n)^4 \cdot \exp(-2p_2 n + (\log n)/15) = o(1)$. Let X_4 be the number of cycles of length at most 3 in $\bar{K}_3$ between vertices in $\bar{S}_2$. By a similar calculation to (13), we have $\mathbb{E}X_4 \le 4 \cdot (4 \log n)^3 \cdot \exp(-p_2 n + (\log n)/30) = o(1)$. Therefore, with high probability $X_3 = X_4 = 0$, and thus $\overline{\mathbf{J5}}$ holds.

$\overline{\mathbf{J6}}$**:** First, let X_5 be the number of paths with length at most 3 between a vertex in $\bar{T}$ and a vertex in $\bar{S}_1$ in $\bar{K}_3$. Similarly to the analysis for (12), we have

$$\begin{aligned}
\mathbb{E}X_5 &\le n(n-1) \cdot \sum_{k=0}^{2} \binom{n-2}{k} 2^{k+1} p_3^{k+1} \cdot 2 \left(\sum_{i=0}^{d} \binom{n-2-k}{i} p_1^i (1-p_1)^{n-2-k-i} \right)^3 \\
&\le 6n \cdot (4 \log n)^3 \cdot \exp(-3p_1 n + (\log n)/10) = o(1).
\end{aligned}$$

Let X_6 be the number of cycles with length at most 3 in $\bar{K}_3$ containing some vertex in $\bar{T}$. Then, similarly to the analysis for (13), we have

$$\begin{aligned}\mathbb{E}X_6 &\leq n \cdot \sum_{k=1}^{2} \binom{n-1}{k} (2p_3)^{k+1} \cdot \left(\sum_{i=0}^{d} \binom{n-1-k}{i} p_1^i (1-p_1)^{n-1-k-i} \right)^2 \\ &\leq 2 \cdot (4\log n)^3 \cdot \exp(-2p_1 n + (\log n)/15) = o(1).\end{aligned}$$

Therefore, with high probability, $X_5 = X_6 = 0$, and hence $\overline{\mathbf{J6}}$ holds.

$\overline{\mathbf{J7}}$**:** Let X_7 be the number of vertices $v \in [n]$ with at least 3 out- or in -neighbours in $\bar{K}_3$ in $\bar{S}_1 \cup N^+_{\bar{K}_3}(\bar{S}_1 - v) \cup N^-_{\bar{K}_3}(\bar{S}_1 - v)$. For each such vertex v, we can pick three different paths, P_1, P_2 and P_3 in $\bar{K}_3$, with length at most 2 which go from v into $\bar{S}_1$. Letting $H = P_1 \cup P_2 \cup P_3$, we have that H is a tree (with maybe some doubled edges) if the endvertices of the paths which are not v are distinct. If $|V(H) \cap \bar{S}_1| = 2$, then we see that H has at least $|H|$ edges, while if $|V(H) \cap \bar{S}_1| = 1$, then H has at least $|H| + 1$ edges. Thus, deleting edges if necessary, there is some digraph $H \subset \bar{K}_3$ and some $j \in [3]$ so that $v \in V(H)$, $j \leq |H| \leq 7$, $e(H) \geq |H| + 2 - j$ edges, and $V(H)$ contains j vertices in $\bar{S}_1$.

Therefore, using Proposition 4.1, we have, for large n, that

$$\begin{aligned}\mathbb{E}X_7 &\leq n \cdot \sum_{j=1}^{3} \sum_{k=j}^{7} \binom{n-1}{k-1} \cdot \binom{k(k-1)}{k+2-j} \cdot p_3^{k+2-j} \cdot \left(\sum_{i=0}^{d} \binom{n-k}{i} p_1^i (1-p_1)^{n-i-k} \right)^j \\ &\leq \sum_{j=1}^{3} \sum_{k=j}^{7} (np_3)^{k+2-j} \cdot n^{j-2} \cdot 2^{k(k-1)} \cdot \exp(-j \cdot p_1 n + j(\log n)/30) \\ &\leq \log^9 n \cdot \sum_{j=1}^{3} n^{j-2} \cdot \exp(-(2j/5)\log n) = o(1).\end{aligned}$$

Therefore, with high probability, $X_7 = 0$, and thus $\overline{\mathbf{J7}}$ holds.

$\overline{\mathbf{J8}}$**:** Let $t_0 = (\log n)^{1/3}/200$, $t_1 = 100n \log\log n / \log n$, $d_0 = (\log n)^{2/3}/2$ and $d_1 = 100(\log n)^{1/3}$. Note that if there are some sets $A, B \subset [n]$ satisfying $|B| < d_1|A|$ and, for some $\diamond \in \{+, -\}$, $d^{\diamond}_{\bar{K}_3}(v, B) \geq d_0$ for each $v \in A$, we have $|B| \geq d_0$, so that $|A| > d_0/d_1 = t_0$. Furthermore, for such a pair A, B, we can add vertices to B to ensure that $|B| = d_1|A|$. Now, there is some $\diamond \in \{+, -\}$ and t with $t_0 \leq t \leq t_1$ and a pair of sets $A, B \subset [n]$ with $|A| = t$, $|B| = d_1 t$, and $d^{\diamond}_{\bar{K}_3}(v, B) \geq d_0$ for each $v \in A$, with probability at most

$$\begin{aligned}2\sum_{t=t_0}^{t_1} \binom{n}{t}\binom{n}{d_1 t}\binom{d_1 t}{d_0}^t p_3^{d_0 t} &\leq 2\sum_{t=t_0}^{t_1} \left(\left(\frac{en}{t}\right) \cdot \left(\frac{en}{d_1 t}\right)^{d_1} \cdot \left(\frac{e d_1 t p_3}{d_0}\right)^{d_0} \right)^t \\ &\leq 2\sum_{t=t_0}^{t_1} \left(\left(\frac{en}{t}\right)^{d_1+1} \cdot \left(\frac{t(\log n)^{3/4}}{n}\right)^{d_0} \right)^t \\ &\leq 2\sum_{t=t_0}^{t_1} \left(e^{d_1+1} \cdot \left(\frac{t}{n}\right)^{d_0-d_1-1} \cdot (\log n)^{3d_0/4} \right)^t \\ &\leq 2\sum_{t=t_0}^{t_1} \left(e^{7d_0/8} \cdot \left(\frac{t}{n}\right)^{7d_0/8} \cdot (\log n)^{3d_0/4} \right)^t \\ &= 2\sum_{t=t_0}^{t_1} \left(\frac{et(\log n)^{6/7}}{n}\right)^{7d_0 t/8} \leq 2\sum_{t=t_0}^{t_1} \left(\frac{1}{2}\right)^{7d_0 t/8} \leq \frac{4}{2^{-7d_0 t_0/8}} = o(1).\end{aligned}$$

Therefore, with high probability, $\overline{\mathbf{J8}}$ holds.

$\overline{\mathbf{J9}}$**:** Let X_8 be the number of vertices in $\bar{K}_3$ with in-degree larger than $50\log n$ or out-degree larger than $50\log n$. Then,

$$\mathbb{E}X_8 \leq 2n\binom{n-1}{50\log n}p_3^{50\log n} \leq 2n\left(\frac{enp_3}{50\log n}\right)^{50\log n} \leq 2n\left(\frac{1}{10}\right)^{50\log n} = o(1).$$

Therefore, with high probability, $\overline{\mathbf{J9}}$ holds. □

We now prove the properties **J10**–**J12** are likely to hold. In the following lemma, the random digraphs do not interact in the properties (nor is K_0 used), but for convenience we use the same distribution for the random digraphs as in Lemma 4.2.

Lemma 4.3. *Let (K_0, K_1, K_2, K_3) be drawn uniformly at random from the set of such tuples for which, for each $j \in \{0, 1, 2, 3\}$, K_j is a digraph with vertex set $[n]$ and i_j edges (as set in (9)), and $K_0 \subset K_1 \subset K_2 \subset K_3$.*

Then, with high probability, the following hold.

J10 *There is some $v \in [n]$ with $d^+_{K_1}(v) + d^-_{K_1}(v) = 0$.*

J11 *The number of vertices in K_2 with in-degree 0 is at least $n^{1/5}$.*

J12 $\delta^\pm(K_3) \geq 2$.

Proof. **J10:** Let $p_1 = (i_1+n)/n(n-1)$. We will construct a random digraph D_1 with the same distribution as $D(n, p_1)$. Similarly as for (11), by Lemma 3.8 we will then have that $\mathbb{P}(e(D_1) \geq i_1) = 1 - o(1)$, and thus, it is sufficient to show that, with high probability, there is some $v \in [n]$ with $d^+_{D_1}(v) + d^-_{D_1}(v) = 0$.

Let then $q = p_1/(4 - 2p_1)$ and $p = p_1(2 - p_1)$, and let $G = G(n, p)$. Form D_1 on the vertex set $[n]$ by taking each edge $uv \in E(G)$ and, independently at random, adding uv but not vu to $E(D_1)$ with probability $(1/2 - q)$, adding vu but not uv to $E(D_1)$ with probability $(1/2 - q)$, and adding both uv and vu to $E(D_1)$ with probability $2q$. Note that, as $p(1/2+q) = p_1$ and $2pq = p_1^2$, D_1 has the same distribution as $D(n, p_1)$. As $p = (\log n - \omega(1))/n$, with high probability $\delta(G) = 0$ (see, for example, Theorems 3.5 and 2.2(ii) in [5]). Furthermore, any $v \in [n]$ with $d_G(v) = 0$ satisfies $d^+_{D_1}(v) = d^-_{D_1}(v) = 0$, and thus **J10** holds with high probability.

J11: Let $p_2 = (i_2 + n)/n(n - 1)$ and $D_2 = D(n, p_2)$. Similarly as for (11), by Lemma 3.8, we have that $\mathbb{P}(e(D_2) \geq i_2) = 1 - o(1)$. Therefore, it is sufficient to show that, with high probability, the number of vertices in D_2 with in-degree 0 is at least $n^{1/5}$. Note that, for each $v \in [n]$, the probability that $d^-_{D_2}(v) = 0$ is $(1-p_2)^{n-1} = \exp(-(1-o(1))p_2 n) = \exp(-(3/4-o(1))\log n)$. Furthermore, this is independent for each $v \in [n]$, and the expected number of vertices in $[n]$ with $d^-_{D_2}(v) = 0$ is $n\cdot\exp(-(3/4-o(1))\log n) = \omega(n^{1/5})$. Therefore, by Lemma 3.8, the probability that the number of vertices in D_2 with in-degree 0 is at least $n^{1/5}$ is $1 - o(1)$. Thus, with high probability, **J11** holds.

J12: Let $p_3 = (i_3 - n)/n(n-1)$, so that $p_3(n-2) = \log n + \log\log n + \omega(1)$. Let $D_3 = D(n, p_3)$. Similarly as for (11), by Lemma 3.8, we have that $\mathbb{P}(e(D_3) \leq i_3) = 1 - o(1)$. Therefore, it is sufficient to show that, with high probability, $\delta^\pm(D_3) \geq 2$.

Let X be the number of vertices $v \in [n]$ with $d^+_{D_3}(v) \leq 1$ or $d^-_{D_3}(v) \leq 1$. Then,

$$\mathbb{E}X \leq 2n \cdot \sum_{i=0}^{1}\binom{n-1}{i}p_3^i(1-p_3)^{n-1-i} \leq 4n \cdot (np_3) \cdot \exp(-p_3(n-2)) = o(1),$$

so that, with high probability $X = 0$, and hence $\delta^\pm(D_3) \geq 2$. Thus, with high probability, **J12** holds. □

4.2 Choosing vertices in the cycle

When embedding each cycle for Theorem 2.4, we have to use the vertices in the cycle with out-degree 0, 2 and 1 to cover vertices in the random digraph D_i with out-degree 0, in-degree 0, and both in- and out-degree 1, respectively. To cover the other low in- or out-degree vertices in D_i, we have more choice. We will select the vertices in the cycle to use with the following lemma, which, furthermore, picks the vertices to be in some linear-sized subpath of the cycle.

Lemma 4.4. *There is some n_0 such that the following holds for each $n \geq n_0$ and $\lambda \in \mathbb{N}$. Suppose C is an oriented n-vertex cycle with λ vertices with out-degree 0. Let $\mu_0, \mu_2 \in \mathbb{N}$ with $\mu_0 + \mu_2 = \lceil 2\lambda/\log n \rceil$ and let $\mu_1 = \lceil (n-2\lambda)/\log n \rceil$. Then, there exists a path $P \subset C$ with length at most $n/100$ and vertex sets $Z_0, Z_1, Z_2 \subset V(P)$ such that the following hold.*

- *The vertices in $Z_0 \cup Z_1 \cup Z_2$ are pairwise at least $100 \log n/\log\log n$ apart on P from each other.*
- *For each $i \in [3]$, $|Z_i| = \mu_i$ and each vertex $v \in Z_i$ has out-degree i in C.*

Proof. For each $i \in \{0,1,2\}$, let $X_i = \{v \in V(C) : d^+_C(v) = i\}$. Let $k = 100 \log n/\log\log n$ and $\ell = \lfloor n/100 \rfloor$. We will first choose the path P using the following claim.

Claim 7. There is a path P with length ℓ such that the following hold.

K1 If $\mu_0 > 0$, then $|V(P) \cap X_0| \geq 1 + (\mu_0 + \mu_2 - 1)\log n/10^3$.

K2 If $\mu_1 > 0$, then $|V(P) \cap X_1| \geq 1 + (\mu_1 - 1)\log n/10^3$.

K3 If $\mu_2 > 0$, then $|V(P) \cap X_2| \geq 1 + (\mu_0 + \mu_2 - 1)\log n/10^3$.

Proof of Claim 7. If $\lambda = 0$ or $n - 2\lambda = 0$, then let P be any path in C with length ℓ. Note that in the first case $\mu_0 + \mu_2 = 0$ and $|V(P) \cap X_1| = |P|$, and thus **K1**–**K3** hold. In the second case, $\mu_1 = 0$ and $|V(P) \cap (X_0 \cup X_2)| = |P|$, and therefore, as any two vertices with out-degree 0 in C on P must have some vertex of in-degree 0 in C between them, we have $|V(P) \cap X_0|, |V(P) \cap X_2| \geq |P|/2 - 1$. Thus, **K1**–**K3** hold in this case.

If $0 < \lambda \leq (\log n)/2$, then $\mu_0 + \mu_2 = 1$, so that either $\mu_0 = 0$ or $\mu_2 = 0$. If the first case occurs, then, using $\lambda > 0$, let P be any path in C with length ℓ with $|V(P) \cap X_2| \geq 1$, and, otherwise, let P be any path in C with length ℓ and $|V(P) \cap X_0| \geq 1$. Note that, in each case, we have $|V(P) \cap X_1| \geq \ell + 1 - 2\lambda \geq n/200 \geq 1 + \mu_1 \log n/10^3$. Thus, we have that **K1**–**K3** hold.

If $0 < n - 2\lambda \leq (\log n)/2$, then $\mu_1 = 1$. Using that $n - 2\lambda > 0$, let P be any path in C with length ℓ with $|V(P) \cap X_1| \geq 1$. Note that we have $|V(P) \cap (X_0 \cup X_2)| \geq \ell + 1 - (n - 2\lambda) \geq n/200$. As any two vertices with out-degree 0 in C on P must have some vertex of in-degree 0 in C between them, we have that $|V(P) \cap X_0|, |V(P) \cap X_2| \geq n/450 \geq \lambda/450 \geq 1 + (\mu_0 + \mu_2)\log n/10^3$. Thus, we have that **K1**–**K3** hold.

Assume then that $\lambda > (\log n)/2$ and $n - 2\lambda > (\log n)/2$. Pick an arbitrary direction of C, and label the vertices of C as $v_1, \ldots, v_n$ in this order. For each $i \in [n]$, let P_i be the path $v_i v_{i+1} \ldots v_{i+\ell}$, with addition modulo n in the indices. For each $i \in [n]$, let $f(i) = \frac{n-2\lambda}{n}(\ell+1) - |V(P_i) \cap X_1|$. Let $f(n+1) = f(1)$. Note that

$$\sum_{i \in [n]} f(i) = (n - 2\lambda)(\ell + 1) - |V(P_i) \cap X_1|(\ell + 1) = 0.$$

As $|f(i) - f(i+1)| \leq 1$ for each $i \in [n]$, we can thus choose $j \in [n]$ with $|f(j)| \leq 1$. Then, as $n - 2\lambda > (\log n)/2$, we have

$$|V(P_j) \cap X_1| \geq \frac{n-2\lambda}{n}(\ell+1) - 1 \geq \frac{n-2\lambda}{100} - 1 \geq 1 + \frac{n-2\lambda}{200} \geq 1 + \frac{(\mu_1 - 1)\log n}{10^3}$$

and, as $|V(P_j) \cap X_1| \leq \frac{n-2\lambda}{n}(\ell+1) + 1 = (\ell+1) - \frac{2\lambda}{n}(\ell+1) + 1$, and $\lambda > (\log n)/2$, we have

$$|V(P_j) \cap (X_0 \cup X_2)| \geq \frac{2\lambda}{n}(\ell+1) - 1 \geq \frac{2\lambda}{100} - 1 \geq 3 + \frac{2\lambda}{200} \geq 3 + \frac{2(\mu_0 + \mu_2 - 1)\log n}{10^3}.$$

As any two vertices in P_j with out-degree 0 on C must have some vertex of in-degree 0 in C between them in P_j, we have that $|V(P_j) \cap X_0|, |V(P_j) \cap X_2| \geq 1 + (\mu_0 + \mu_2 - 1)\log n/10^3$. Let $P = P_j$. In every case, we have now chosen a path P with length ℓ such that **K1**–**K3** hold. □

Given the path P as in Claim 7, we now pick the sets Z_0, Z_1 and Z_2. We do this in two cases, according to whether $\mu_0 + \mu_2 \leq \mu_1$ or $\mu_0 + \mu_2 > \mu_1$. In each case, we pick the smaller of $Z_0 \cup Z_2$ and Z_1 first.

Case I. Suppose that $\mu_0 + \mu_2 \leq \mu_1$. Pick vertex sets $Z_0 \subset X_0$ and $Z_2 \subset X_2$ so that the vertices in $Z_0 \cup Z_2$ are pairwise at least k apart on P, $|Z_0| \leq \mu_0$, $|Z_2| \leq \mu_2$, and, subject to this, $|Z_0 \cup Z_2|$ is maximised. Suppose, for contradiction, that $|Z_0| < \mu_0$. If $|Z_0| = 0$ and $\mu_2 = 0$, then $Z_2 = \emptyset$ and, by **K1**, we have $|V(P) \cap X_0| \geq 1$, so that there is a vertex $z \in V(P) \cap X_0$ which is a distance at least k apart from every vertex in $Z_1 \cup Z_2 = \emptyset$ on P, a contradiction. Therefore, we can assume that $|Z_0| > 0$ or $\mu_2 > 0$. In each case, we have, as $\mu_0 > |Z_0|$, that $\mu_0 + \mu_2 \geq 2$. Hence, by **K1**, $|V(P) \cap X_0| \geq (\mu_0 + \mu_2) \log n/(2 \cdot 10^3)$. Now, every vertex in $V(P) \cap X_0$ is within distance $k-1$ of some vertex in $Z_0 \cup Z_2$ on P, so that

$$|V(P) \cap X_0| \leq (2k-1)|Z_0 \cup Z_2| < (2k-1)(\mu_0 + \mu_2) \leq \frac{10^3(\mu_0 + \mu_2) \log n}{\log \log n}.$$

For sufficiently large n this contradicts $|V(P) \cap X_0| \geq (\mu_0 + \mu_2) \log n/(2 \cdot 10^3)$. Similarly, we get a contradiction if $|Z_2| < \mu_2$. Therefore, we have $|Z_0| = \mu_0$ and $|Z_2| = \mu_2$.

Let then $Z_1 \subset X_1$ be a maximal set subject to $|Z_1| \leq \mu_1$ and that the vertices in $Z_0 \cup Z_1 \cup Z_2$ are pairwise at least k apart on P. Suppose, for contradiction, that $|Z_1| < \mu_1$. Then, every vertex in X_1 is within distance $k-1$ of some vertex in $Z_0 \cup Z_1 \cup Z_2$ on P, so that

$$|V(P) \cap X_1| \leq (2k-1)|Z_0 \cup Z_1 \cup Z_2| < (2k-1)(\mu_0 + \mu_1 + \mu_2) \leq (2k-1) \cdot 2\mu_1 \leq \frac{10^3 \mu_1 \log n}{\log \log n}.$$

As $\mu_1 \geq \mu_0 + \mu_2$ (and $\mu_0 + \mu_1 + \mu_2 \geq n/\log n$), we have $\mu_1 \geq 2$. Therefore, for sufficiently large n, this contradicts **K2**. Thus, we have $|Z_1| = \mu_1$, and Z_0, Z_1, Z_2 and P satisfy the conditions in the lemma.

Case II. Assume then that $\mu_0 + \mu_2 > \mu_1$. Let $Z_1 \subset X_1$ be a maximal set subject to $|Z_1| \leq \mu_1$ and that the vertices in Z_1 are pairwise at least k apart on P. Suppose, for contradiction, that $|Z_1| < \mu_1$. If $|Z_1| = 0$, then $\mu_1 > 0$ and, by **K2**, we have $|V(P) \cap X_1| \geq 1$, so that there is a vertex $z \in V(P) \cap X_1$ which is a distance at least k apart from every vertex in $Z_1 = \emptyset$, a contradiction. Therefore, we can assume that $\mu_1 > |Z_1| > 0$, and hence, by **K2**, that $|V(P) \cap X_1| \geq \mu_1 \log n/(2 \cdot 10^3)$. Then, every vertex in X_1 is within distance $2k-1$ of Z_1 on P, so that

$$|V(P) \cap X_1| \leq (2k-1)|Z_1| < (2k-1)\mu_1 \leq \frac{10^3 \mu_1 \log n}{\log \log n}.$$

For sufficiently large n this contradicts $|V(P) \cap X_1| \geq \mu_1 \log n/(2 \cdot 10^3)$. Therefore, we have $|Z_1| = \mu_1$.

Pick vertex sets $Z_0 \subset X_0$ and $Z_2 \subset X_2$ so that the vertices in $Z_0 \cup Z_1 \cup Z_2$ are pairwise at least k apart on P, $|Z_0| \leq \mu_0$, and $|Z_2| \leq \mu_2$, and, subject to this, $|Z_0 \cup Z_2|$ is maximised. Suppose, for contradiction, that $|Z_0| < \mu_0$. Then, as every vertex in $V(P) \cap X_0$ is within distance $k-1$ of $Z_0 \cup Z_1 \cup Z_2$ on P, and $\mu_0 + \mu_2 > \mu_1$, we have

$$|V(P) \cap X_0| \leq (2k-1)|Z_0 \cup Z_1 \cup Z_2| < (2k-1)(\mu_0 + \mu_1 + \mu_2) \leq \frac{10^3(\mu_0 + \mu_2) \log n}{\log \log n}. \tag{14}$$

As $\mu_0 + \mu_2 > \mu_1$ (and $\mu_0 + \mu_1 + \mu_2 \geq n/\log n$), we have $\mu_0 + \mu_2 \geq 2$. Thus, by **K1**, $|V(P) \cap X_0| \geq (\mu_0 + \mu_2) \log n/(2 \cdot 10^3)$, which, for large n, contradicts (14). Similarly, we get a contradiction if $|Z_2| < \mu_2$. Therefore, we have $|Z_0| = \mu_0$, $|Z_2| = \mu_2$, and thus Z_0, Z_1, Z_2 and P satisfy the conditions in the lemma. □

4.3 Proof of Theorem 2.4 from Theorem 2.3

We can now prove Theorem 2.4. In the initial set-up of the proof, we follow the explanation at the start of Section 4.1, and use the values of i_0, i_1, i_2 and i_3 in (9). In the random digraph process $D_0, D_1, \ldots, D_{n(n-1)}$, we condition on the value of D_{i_0}, and, for each i with $i_0 \leq i \leq i_3$, on the edges in $E(D_i) \setminus E(D_{i_0})$ with at least one vertex in S, where S is the set of low in- or out-degree vertices in D_{i_0}. For each i with $i_1 \leq i \leq i_2$, subject to this conditioning, we know which cycles we want to embed into D_i (gathered into the set $\mathcal{C}_i$). The main part of the proof consists of showing that, subject to the conditioning, a copy of each such cycle C is very likely to appear in D_i (see Claim 8). Taking a union bound then completes the proof.

As highlighted by paragraph titles, the proof that a relevant cycle C is very likely to appear in D_i subject to the conditioning (that is, the proof of Claim 8) proceeds in the following steps.

- We simplify the random edges in D_i which have not been conditioned on, replacing them with a binomial random digraph with vertex set $[n] \setminus S$ (called $\bar{D}$).
- We identify the low in- and out-degree vertices in D_i (which form a subset of S), and partition them according to the degree of the vertex in the cycle which will be embedded to them, before choosing vertices to embed as their neighbours to get paths of length 2.
- We choose the subpath P of C so that we can embed well-spaced paths of length 2 from P to these paths of length 2 in D_i (using Lemma 4.4).
- We modify the conditioned edges of D_i (those in the graph H_i) to get a pseudorandom digraph (H_i'), by contracting the chosen paths of length 2 in D_i (and possibly altering some edges).
- We modify C and P accordingly.
- We modify the binomial random digraph $\bar{D}$ with vertex set $[n] \setminus S$ accordingly.
- We apply Theorem 2.3 to these modified digraphs, before undoing the modifications to find a copy of the cycle C in $D_i \cup \bar{D}$.

Proof of Theorem 2.4. Let $d = \log n/300$ and let i_0, i_1, i_2 and i_3 be as in (9). Let $D_0, D_1, \ldots, D_{n(n-1)}$ be the n-vertex random digraph process. First, note that, by **J10** in Lemma 4.2, with high probability there is some $v \in [n]$ with $d^+_{D_{i_1}}(v) = d^-_{D_{i_1}}(v) = 0$, and thus, for each $0 \le i \le i_1$ the property in Theorem 2.4 for D_i holds trivially. Similarly, by **J12** in Lemma 4.2, we have, with high probability that $\delta^\pm(D_{i_3}) \ge 2$, and hence $s_{i_3} = t_{i_3} = 0$, and thus we need to show that, with high probability, D_{i_3} contains a copy of every n-vertex oriented cycle. If this holds, then D_i contains a copy of every n-vertex oriented cycle for each $i \ge i_3$. Therefore, it is sufficient show that, with high probability, the property in Theorem 2.4 holds for each $i_1 \le i \le i_3$.

Let

$$S = \{v \in [n] : d^+_{D_{i_0}}(v) \le d \text{ or } d^-_{D_{i_0}}(v) \le d\}.$$

For each $i_0 \le i \le i_3$, let D_i' be the (random) digraph with vertex set $[n]$ and edge set

$$E(D_{i_0}) \cup \{xy \in E(D_i) : \{x, y\} \cap S \neq \emptyset\},$$

so that $D_i' \subset D_i$, and $E(D_i) \setminus E(D_i') \subset ([n] \setminus S) \times ([n] \setminus S)$.

Now, for each $j \in \{0, 1, 2, 3\}$, let $K_j = D_{i_j}$, $\bar{K}_j = D'_{i_j}$,

$$S_j = \{v \in [n] : d^+_{K_j}(v) \le d \text{ or } d^-_{K_j}(v) \le d\} = \{v \in [n] : d^+_{\bar{K}_j}(v) \le d \text{ or } d^-_{\bar{K}_j}(v) \le d\},$$

where we have used that $S_0 = S$ and $d^\diamond_{K_j}(v) = d^\diamond_{\bar{K}_j}(v)$ for each $v \in S$, $\diamond \in \{+, -\}$ and $j \in \{0, 1, 2, 3\}$. Furthermore, let

$$T = \{v \in [n] : d^+_{K_1}(v) \le d \text{ and } d^-_{K_1}(v) \le d\} = \{v \in [n] : d^+_{\bar{K}_1}(v) \le d \text{ and } d^-_{\bar{K}_1}(v) \le d\},$$

and let $\overline{\textbf{J1}}$–$\overline{\textbf{J9}}$ and $\overline{\textbf{J11}}$ be the properties **J2**–**J9** and **J11** with K_j replaced by $\bar{K}_j$ for any relevant $j \in \{0, 1, 2, 3\}$. As $\bar{K}_0 = K_0$ and $\bar{K}_3 \subset K_3$, each of **J3**–**J9** and **J11** implies the corresponding property $\overline{\textbf{J3}}$–$\overline{\textbf{J9}}$ and $\overline{\textbf{J11}}$. Therefore, by Lemma 4.2, we have that, with high probability, $\overline{\textbf{J1}}$–$\overline{\textbf{J9}}$ and $\overline{\textbf{J11}}$ hold. Let $\mathcal{D} = (D'_{i_0}, D'_{i_0+1}, \ldots, D'_{i_3})$. Let $\mathbf{H}$ be the set of possible values of $\mathcal{D}$ for which $\overline{\textbf{J1}}$–$\overline{\textbf{J9}}$ and $\overline{\textbf{J11}}$ hold, so that $\mathbb{P}(\mathcal{D} \in \mathbf{H}) = 1 - o(1)$.

Let $\mathcal{C}$ be the set of all oriented cycles whose underlying cycle is the canonical cycle with vertex set $[n]$. For each $i_1 \le i \le i_3$, let s_i be the number of vertices in D_i with in-degree or out-degree 0 and let t_i be the number of vertices in D_i with in-degree 1 and out-degree 1. Let $\mathcal{C}_i$ be the set of cycles in $\mathcal{C}$ with at least $1 + (s_i - 1)\log n$ and at most $n - 1 - (t_i - 1)\log n$ changes of direction. We will show the following claim.

Claim 8. There is some n_0 such that, if $n \ge n_0$, then, for each $\mathcal{H} \in \mathbf{H}$, $i_1 \le i \le i_3$, and $C \in \mathcal{C}$,

$$\mathbb{P}(d^+_{D_i}(v) + d^-_{D_i}(v) \ge 2 \text{ for each } v \in [n], \text{ and } C \in \mathcal{C}_i, \text{ and } C \not\subseteq D_i | \mathcal{D} = \mathcal{H}) \le 2e^{-n}. \tag{15}$$

This claim is sufficient to prove the theorem. Indeed, let E be the event that, for some $i_1 \leq i \leq i_3$ and $C \in \mathcal{C}$, we have $d^+_{H_i}(v) + d^-_{H_i}(v) \geq 2$ for each $v \in [n]$ and $C \in \mathcal{C}_i$ and $C \not\subseteq D_i$. Then, if $n \geq n_0$, by a simple union bound and Claim 8, we have, for each $\mathcal{H} \in \mathbf{H}$, that $\mathbb{P}(E|\mathcal{D} = \mathcal{H}) \leq (i_3 - i_1 + 1) \cdot 2^n \cdot 2e^{-n} \leq n^2 \cdot 2^n \cdot 2e^{-n}$ if $n \geq n_0$, and hence

$$
\begin{aligned}
\mathbb{P}(E) &\leq \mathbb{P}(\mathcal{D} \notin \mathbf{H}) + \sum_{\mathcal{H} \in \mathbf{H}} \mathbb{P}(\mathcal{D} = \mathcal{H}) \cdot \mathbb{P}(E|\mathcal{D} = \mathcal{H}) \\
&\leq \mathbb{P}(\mathcal{D} \notin \mathbf{H}) + n^2 \cdot 2^n \cdot 2e^{-n} \cdot \sum_{\mathcal{H} \in \mathbf{H}} \mathbb{P}(\mathcal{D} = \mathcal{H}) \leq \mathbb{P}(\mathcal{D} \notin \mathbf{H}) + n^2 \cdot 2^n \cdot e^{-n} = \mathbb{P}(\mathcal{D} \notin \mathbf{H}) + o(1).
\end{aligned}
$$

Thus, as $\mathbb{P}(\mathcal{D} \notin \mathbf{H}) = o(1)$, we have that, with high probability, E does not hold, and therefore the property in the theorem holds. It is left then to prove Claim 8.

Proof of Claim 8. Let $\mathcal{H} = (H_{i_0}, \ldots, H_{i_3}) \in \mathbf{H}$, $i_1 \leq i \leq i_3$ and $C \in \mathcal{C}$. Let s be the number of in-degree 0 or out-degree 0 vertices in H_i and let t be the number of vertices in H_i with in- and out-degree 1. For each $v \in [n]$, if $\mathcal{D} = \mathcal{H}$ and either $d^+_{H_i}(v) \leq d$ or $d^-_{H_i}(v) \leq d$, we have, as $D_{i_0} = D'_{i_0} = H_{i_0}$, that $v \in S_0$ and hence $d^\diamond_{D_i}(v) = d^\diamond_{H_i}(v)$ for each $\diamond \in \{+, -\}$. Therefore, if $\mathcal{D} = \mathcal{H}$, then $s_i = s$ and $t_i = t$. Thus, given $\mathcal{D} = \mathcal{H}$, whether or not $C \in \mathcal{C}_i$ depends only on $\mathcal{H}$ and i, so we can assume that $C \in \mathcal{C}_i$, as otherwise (15) holds trivially. Similarly, if $d^+_{H_i}(v) + d^-_{H_i}(v) \leq 1$ for some $v \in [n]$, then $d^+_{D_i}(v) + d^-_{D_i}(v) \leq 1$ and (15) again trivially holds. Thus, we can assume that $d^+_{H_i}(v) + d^-_{H_i}(v) \geq 2$ for each $v \in [n]$.

Given $\mathcal{D} = \mathcal{H}$, we have that $S = S_0$ is the fixed set $\{v \in [n] : d^+_{H_{i_0}}(v) \leq d \text{ or } d^-_{H_{i_0}}(v) \leq d\}$, which depends only on $\mathcal{H}$. Therefore, conditioned on $\mathcal{D} = \mathcal{H}$, the distribution of the random digraph D_i is that of the deterministic graph H_i with $i - e(H_i)$ edges uniformly at random added from $\{uv \notin E(H_i) : u, v \in [n] \setminus S, u \neq v\}$. We will first replace these random edges with an appropriate binomial random digraph.

Simplify the random edges within $[n] \setminus S$. Let $p = \log n / 100n$, and let $\bar{D}$ be a binomial random digraph with edge probability p and vertex set $[n] \setminus S$. Let E' be the event that $e(\bar{D}) \leq i - e(H_i)$. If E' holds, then let $\hat{D}$ be $\bar{D}$ with $i - e(H_i) - |E(\bar{D}) \setminus E(H_i)|$ edges uniformly at random added from $\{uv \notin E(H_i) \cup E(\bar{D}) : u, v \in [n] \setminus S, u \neq v\}$. If E' does not hold, then let $\hat{D}$ be the random digraph with vertex set $[n] \setminus S$ with $i - e(H_i)$ edges uniformly at random added from $\{uv \notin E(H_i) : u, v \in [n] \setminus S, u \neq v\}$. Note that, $H_i \cup \hat{D}$ has the same distribution as D_i conditioned on $\mathcal{D} = \mathcal{H}$. Furthermore, if E' holds, then $H_i \cup \bar{D} \subset H_i \cup \hat{D}$.

As $\mathcal{H} \in \mathbf{H}$, we have that $\overline{\mathbf{J2}}$ holds whenever $\mathcal{D} = \mathcal{H}$, and therefore $i - e(H_i) \geq i_1 - e(H_{i_1}) \geq i_1 - i_0 - n \geq n \log n / 40$, for sufficiently large n. Thus, as $\mathbb{E}(e(\bar{D})) \leq p(n - |S|)^2 \leq pn^2$, by Lemma 3.8, we have $\mathbb{P}(E' \text{ does not hold}) = \exp(-\Omega(n \log n)) \leq \exp(-n)$, for sufficiently large n.

Thus, to complete the proof of the claim, it is sufficient to show that, for sufficiently large n, $\mathbb{P}(C \not\subseteq H_i \cup \bar{D}) \leq \exp(-n)$, for it follows that

$$
\mathbb{P}(C \not\subseteq D_i | \mathcal{D} = \mathcal{H}) = \mathbb{P}(C \not\subseteq H_i \cup \hat{D}) \leq \mathbb{P}(E' \text{ does not hold}) + \mathbb{P}(C \not\subseteq H_i \cup \bar{D}) \leq 2 \cdot \exp(-n).
$$

We now focus on the vertices in H_i with low in- or out-degree.

Identify low in- and out-degree vertices. Recalling that $d = \log n / 300$, let

$$
Y = \{v \in [n] : d^+_{H_i}(v) \leq d \;\text{ or }\; d^-_{H_i}(v) \leq d\},
$$

so that $Y \subset S_1$. For each $j \in \{0, 1, 2\}$, let $Y_j \subset Y$ be the vertices in Y which, based on their out- and in-degree, could be a vertex of the copy of C in H_i with out-degree j. That is, let

$$
Y_0 = \{v \in Y : d^-_{H_i}(v) \geq 2\}, \;\; Y_1 = \{v \in Y : d^\pm_{H_i}(v) \geq 1\}, \quad \text{and} \quad Y_2 = \{v \in Y : d^+_{H_i}(v) \geq 2\}.
$$

As $d^-_{H_i}(v) + d^+_{H_i}(v) \geq 2$ for each $v \in [n]$, we have $Y = Y_0 \cup Y_1 \cup Y_2$. For each $\diamond \in \{+, -\}$, we will let $Y^\diamond \subset Y$ be the set of vertices with plenty of $\diamond$-neighbours in H_i, so that it is the set of vertices which are easy to place in the copy of C as a vertex with two $\diamond$-neighbours. That is, we let, for each $\diamond \in \{+, -\}$,

$$
Y^\diamond = \{v \in Y : d^\diamond_{H_i}(v) > d\},
$$

so that, as $Y \subset S_1$, Y^+ and Y^- are disjoint. Recall that T is the set of vertices in $H_{i_1} \subset H_i$ with in- and out-degree both at most d, so that $Y = (Y \cap T) \cup Y^+ \cup Y^-$.

Our aim now is to partition Y into sets $\bar{Y}_j$, $j \in \{0,1,2\}$, so that $|\bar{Y}_j|$, $j \in \{0,1,2\}$, satisfy certain inequalities for a later application of Lemma 4.4. We will find them with vertices $x_v, y_v \in [n] \setminus Y$, for each $v \in Y$, where x_v and y_v can be the neighbours of $v \in \bar{Y}_j$ on any cycle in which v has out degree j. We do this differently according to the number of changes of direction on C. Let λ then be the number of vertices of C with out-degree 0. As $C \in \mathcal{C}_i$, we have $1 + (s_i - 1)\log n \leq 2\lambda \leq n - 1 - (t_i - 1)\log n$. Our cases are when $\lambda \geq n/4$ and when $\lambda < n/4$.

Case I: when $\lambda \geq n/4$. There are many changes of direction on C, so we will use vertices with out-degree 0 or 2 in C to cover as many vertices as possible. Thus, let $\bar{Y}_0 = (Y \cap Y^-) \cup (Y_0 \cap T)$, $\bar{Y}_2 = (Y \cap Y^+) \cup ((Y_2 \cap T) \setminus \bar{Y}_0)$, and $\bar{Y}_1 = Y \setminus (\bar{Y}_0 \cup \bar{Y}_2)$. Note that each vertex in $\bar{Y}_1$ has in- and out-degree exactly 1 in H_i. Thus, $|\bar{Y}_1| = t_i$ and $\bar{Y}_1 \subset T$. As $2\lambda \leq n - 1 - (t_i - 1)\log n$, we have $n - 2\lambda \geq 1 + (t_i - 1)\log n$, so that $\lceil (n - 2\lambda)/\log n \rceil \geq t_i = |\bar{Y}_1|$. As $\overline{\mathbf{J1}}$ holds if $\mathcal{D} = \mathcal{H}$, we have that $|S_0| \leq n^{2/3}$, and thus $|\bar{Y}_0| + |\bar{Y}_2| \leq |Y| \leq |S_0| \leq 2\lambda/\log n$.

Now, for each $v \in \bar{Y}_0 \cap T \subset Y_0$, pick distinct $x_v, y_v \in N^-_{H_i}(v)$. For each $v \in \bar{Y}_1 \subset T$, pick $x_v \in N^+_{H_i}(v)$ and $y_v \in N^-_{H_i}(v)$. For each $v \in \bar{Y}_2 \cap T \subset Y_2$, pick distinct $x_v, y_v \in N^+_{H_i}(v)$. As $\mathcal{H} \in \mathbf{H}$, and $\overline{\mathbf{J6}}$ holds whenever $\mathcal{D} = \mathcal{H}$, we have that $H_i[\{v, x_v, y_v : v \in Y \cap T\}]$ is a forest with $|Y \cap T|$ components and no vertices in $S_1 \setminus (Y \cap T)$, and hence the vertices x_y, y_v, $v \in Y \cap T$, are distinct and in $[n] \setminus S_1 \subset [n] \setminus Y$.

As $\mathcal{H} \in \mathbf{H}$, and $\overline{\mathbf{J7}}$ holds whenever $\mathcal{D} = \mathcal{H}$, and as $H_i \subset K_3$ and $Y \subset S_1$, we have that each $v \in Y$ has at most 2 in- or out-neighbours in H_i in $Y \cup N^+_{H_i}(Y - v) \cup N^-_{H_i}(Y - v)$. Therefore, for each $v \in \bar{Y}_0 \setminus T \subset Y^-$, as $|N^-_{H_i}(v)| \geq d \geq 4$, we can do the following.

L1 Pick distinct $x_v, y_v \in N^-_{H_i}(v) \setminus (Y \cup N^+(Y - v) \cup N^-(Y - v))$.

Similarly, for each $v \in \bar{Y}_2 \setminus T \subset Y^+$, we can do the following.

L2 Pick distinct $x_v, y_v \in N^+_{H_i}(v) \setminus (Y \cup N^+(Y - v) \cup N^-(Y - v))$.

Note that the vertices x_v, y_v, $v \in Y$, are distinct and in $[n] \setminus Y$.

To recap, we have found a partition $Y = \bar{Y}_0 \cup \bar{Y}_1 \cup \bar{Y}_2$, and distinct vertices x_v, y_v, $v \in Y$, in $[n] \setminus Y$, such that the following hold.

M1 If $v \in \bar{Y}_0$, then $x_v, y_v \in N^-_{H_i}(v)$. If $v \in \bar{Y}_1$, then $x_v \in N^+_{H_i}(v)$ and $y_v \in N^-_{H_i}(v)$. If $v \in \bar{Y}_2$, then $x_v, y_v \in N^+_{H_i}(v)$.

M2 $|\bar{Y}_0| + |\bar{Y}_2| \leq \lceil 2\lambda/\log n \rceil$ and $|\bar{Y}_1| \leq \lceil (n - 2\lambda)/\log n \rceil$.

Case II: when $\lambda < n/4$. Supposing then that $\lambda < n/4$, we now find a partition $Y = \bar{Y}_0 \cup \bar{Y}_1 \cup \bar{Y}_2$, and distinct vertices x_v, y_v, $v \in Y$, in $[n] \setminus Y$, which also satisfy **M1** and **M2**. This is more complicated than in Case I, as to achieve **M2** we may have to assign vertices in $Y \setminus T$ not to $\bar{Y}_0 \cup \bar{Y}_2$ but to $\bar{Y}_1$. Thus, all the vertices x_v, y_v, $v \in Y \setminus T$, are not selected in either **L1** or **L2**. To cope with this, we gather into a set B the vertices $v \in Y \setminus T$ for which x_v, y_v would be particularly hard to find were v assigned to $\bar{Y}_1$. We then show there are enough changes of direction in C to assign the vertices in B instead to $\bar{Y}_0$ or $\bar{Y}_2$.

As $\overline{\mathbf{J6}}$ holds whenever $\mathcal{D} = \mathcal{H}$, we can take $B \subset Y \setminus T$ to be a minimal set of vertices for which we have that $H_i[(Y \setminus B) \cup N^+_{H_i}(Y \setminus B) \cup N^-_{H_i}(Y \setminus B)]$ is a forest with $|Y \setminus B|$ components and no vertices in B. For each $\diamond \in \{+, -\}$, let $B^\diamond = \{v \in B : d^\diamond_{H_i}(v) > d\}$. Note that, as $B \subset Y \subset S_1$, B^+ and B^- are disjoint and, as $B \cap T = \emptyset$, $B = B^+ \cup B^-$. Let $\bar{Y}_0 = \{y \in Y : d^+_{H_i}(y) = 0\} \cup B^-$. Let $\bar{Y}_2 = \{y \in Y : d^-_{H_i}(y) = 0\} \cup B^+$. Let $\bar{Y}_1 = Y \setminus (\bar{Y}_1 \cup \bar{Y}_2) \subset Y \setminus B$. As $\overline{\mathbf{J1}}$ holds if $\mathcal{D} = \mathcal{H}$, we have that $|S_0| \leq n^{2/3}$, so that, as $\lambda < n/4$, we have $|\bar{Y}_1| \leq |Y| \leq |S_0| \leq n^{2/3} \leq (n - 2\lambda)/\log n$.

If $i \geq i_2$, then $Y \subset S_2$ and, as $\overline{\mathbf{J5}}$ holds whenever $\mathcal{D} = \mathcal{H}$, and $H_{i_2} \subset H_i$, we have that $B = \emptyset$. Hence, $|\bar{Y}_0| + |\bar{Y}_2| = s_i$, so that, as, $2\lambda \geq 1 + (s_i - 1)\log n$, we have $|\bar{Y}_0| + |\bar{Y}_2| = s_i \leq \lceil 2\lambda/\log n \rceil$.

If $i \leq i_2$, then, as $\overline{\mathbf{J11}}$ holds whenever $\mathcal{D} = \mathcal{H}$, and $H_i \subset H_{i_2}$, we have that $s_i \geq n^{1/5}$. As $\overline{\mathbf{J3}}$ and $\overline{\mathbf{J4}}$ hold whenever $\mathcal{D} = \mathcal{H}$, we have that $|B| \leq 5n^{1/6} \leq s_i - 2$. Thus, as $\lambda \geq 1 + (s_i - 1)\log n$, we have $|\bar{Y}_0| + |\bar{Y}_2| \leq s_i + |B| \leq 2s_i - 2 \leq 2\lambda/\log n$.

Therefore, **M2** holds whether $i \geq i_2$ or $i \leq i_2$. Now, for each $v \in \bar{Y}_0 \setminus B^- \subset Y_0$, pick distinct $x_v, y_v \in N^-_{H_i}(v)$. For each $v \in \bar{Y}_1 \subset Y_1$, pick $x_v \in N^+_{H_i}(v)$ and $y_v \in N^-_{H_i}(v)$. For each $v \in \bar{Y}_2 \setminus B^+ \subset Y_2$, pick distinct $x_v, y_v \in N^+_{H_i}(v)$. As $H_i[\{v, x_v, y_v : v \in Y \setminus B\}] \subset H_i[(Y \setminus B) \cup N^+_{H_i}(Y \setminus B) \cup N^-_{H_i}(Y \setminus B)]$ is a forest with $|Y \setminus B|$ components with no vertices in B, the vertices x_y, y_v, $v \in Y \setminus B$, are distinct, and in $[n] \setminus Y$.

Similarly to in Case I, we have that each $v \in Y$ has at most 2 in- or out-neighbours in $Y \cup N^+_{H_i}(Y - v) \cup N^-_{H_i}(Y - v)$. Therefore, for each $v \in B^-$, as $|N^-_{H_i}(v)| \geq d \geq 4$, we can pick distinct $x_v, y_v \in N^-_{H_i}(v) \setminus (Y \cup N^+_{H_i}(Y - v) \cup N^-_{H_i}(Y - v))$. Similarly, for each $v \in B^+$, we can pick distinct $x_v, y_v \in N^+_{H_i}(v) \setminus (Y \cup N^+_{H_i}(Y - v) \cup N^-_{H_i}(Y - v))$. Note that all the vertices x_v, y_v, $v \in Y$, are distinct and in $[n] \setminus Y$, and therefore **M1** holds.

We have now chosen, in both Case I and Case II, a partition $Y = \bar{Y}_0 \cup \bar{Y}_1 \cup \bar{Y}_2$, and distinct vertices x_v, y_v, $v \in Y$, in $[n] \setminus Y$, satisfying **M1** and **M2**. We now choose the section of the cycle, and vertex sets in it, that we use to cover the low in- and out-degree vertices (those in $Y = \bar{Y}_0 \cup \bar{Y}_1 \cup \bar{Y}_2$).

Choosing P and Z_0, Z_1, Z_2. Now, by Lemma 4.4 and **M2**, there exists a path $P \subset C$ with length at most $n/100$ and vertex sets $Z_0, Z_1, Z_2 \subset V(P)$ such that the following hold with $Z = Z_0 \cup Z_1 \cup Z_2$.

N1 For each $j \in \{0, 1, 2\}$, $|Z_j| = |\bar{Y}_j|$, and the vertices in Z_j each have out-degree j in C.

N2 The vertices in Z are pairwise at least $100 \log n / \log\log n$ apart on P from each other.

Using **N1**, label the vertices of Z as a_v, $v \in Y$, so that, for each $j \in \{0, 1, 2\}$, if $v \in \bar{Y}_j$, then $a_v \in Z_j$. Pick an arbitrary direction on C to be clockwise and, for each $v \in Y$, label vertices of C so that $d_v b_v a_v c_v e_v$ is a path on C with vertices in clockwise order. Using **M1** and **N1**, for each $v \in Y$, by swapping the labels of x_v and y_v if necessary, we can assume that $x_v v y_v$ is a copy of $b_v a_v c_v$.

We now modify H_i, C (along with P) and $\bar{D}$, to allow us to apply Theorem 2.3.

Modify H_i to get a pseudorandom graph, H_i'. For each $v \in Y$, we wish to delete v, x_v and y_v from H_i and replace them with a single new vertex z_v. Later we will find a cycle C_0 containing a subpath through z_v, say with vertices w, z_v, w', and then replace $w z_v w'$ with $w x_v v y_v w'$ to get a copy of the subpath $d_v b_v a_v c_v e_v$ of P. To do this, we use the in-edges of z_v to guarantee the appropriate edge between w and x_v (matching the edge between d_v and b_v) and the out-edges of z_v to guarantee the appropriate edge between y_v and w' (matching the edge between c_v and e_v), and insist later that the subpath on w, z_v, w' in C_0 is directed from w to w'.

More precisely, define H_i' as follows. Let $Y' := \{v, x_v, y_v : v \in Y\}$ and let H_i' be the graph formed by deleting Y' from H_i and adding the new vertices z_v, $v \in Y$, along with the following additional edges for each $v \in Y$ and $w \in [n] \setminus Y'$.

O1 $w z_v$ if $d_v b_v \in E(C)$ and $w x_v \in E(H_i)$

O2 $w z_v$ if $b_v d_v \in E(C)$ and $x_v w \in E(H_i)$

O3 $z_v w$ if $c_v e_v \in E(C)$ and $y_v w \in E(H_i)$

O4 $z_v w$ if $e_v c_v \in E(C)$ and $w y_v \in E(H_i)$

Let $X = \{z_v : v \in Y\}$. Note that $|X| = |Y| \leq n^{2/3}$. Let $\bar{n} = |H_i'| = n - 2|X| \geq n - 2n^{2/3}$, so that, for large n, $|X| \leq \bar{n}^{3/4}$. We will show that H_i' is pseudorandom with exceptional set X. As $\overline{\mathbf{J9}}$ holds whenever $\mathcal{D} = \mathcal{H}$, and $H_i \subset K_{i_3}$, we have that $\Delta(H_i') \leq \Delta(H) \leq 50 n \log n \leq 100 \bar{n} \log \bar{n}$, and therefore **A1** holds for $D = H_i'$.

As $\overline{\mathbf{J7}}$ holds whenever $\mathcal{D} = \mathcal{H}$, we have that any vertex in $[n]$ has at most 2 in- or out-neighbours in H_i in $S_1 \cup N^+_{H_i}(S_1) \cup N^-_{H_i}(S_1)$, and hence at most 2 in- or out-neighbours in H_i in Y'. Thus, for each $v \in [n] \setminus Y' \subset [n] \setminus Y$ and $\diamond \in \{+, -\}$, we have

$$d^{\diamond}_{H_i'}(v, V(H_i') \setminus X) = d^{\diamond}_{H_i}(v, [n] \setminus Y') \geq d - 2 \geq \log n/500 \geq \log \bar{n}/500.$$

For each $v \in Y$, we have that if $d_v b_v \in E(C)$, then $d^-_{H_i}(x_v, [n] \setminus Y')$ in-edges to z_v are added at **O1**, and if $b_v d_v \in E(C)$, then $d^+_{H_i}(x_v, [n] \setminus Y')$ in-edges to z_v are added at **O2**. Therefore, as $x_v \notin Y$,

$$d^-_{H_i'}(z_v, V(H_i') \setminus X) \geq \min\{d^-_{H_i}(x_v, [n] \setminus Y'), d^+_{H_i}(x_v, [n] \setminus Y')\} \geq d - 2 \geq \log n/500 \geq \log \bar{n}/500.$$

Similarly, enough out-edges from z_v are added at either **O3** or **O4** that $d^+_{H'_i}(z_v, V(H'_i) \setminus X) \geq \log \bar{n}/500$. Therefore, **A2** holds for $D = H'_i$ and X.

We now prove that **A3** holds for $D = H'_i$. Suppose, for contradiction, that there are sets $A', B' \subset V(H'_i)$ and some $\diamond \in \{+,-\}$ with $d^\diamond_{H'_i}(v, B') \geq (\log \bar{n})^{2/3}$ for each $v \in A'$, $|B'| \leq |A'|(\log \bar{n})^{1/3}$ and $|A'| \leq \bar{n} \log\log \bar{n}/\log \bar{n}$. Now, every vertex in $v \in [n] \setminus Y'$ has at most 2 in- or out-neighbours in H_i in Y', and hence at most 2 in- or out-neighbours in H'_i in X. Therefore, every vertex in $A' \setminus X$ has at least $(\log \bar{n})^{2/3} - 2 \geq (\log n)^{2/3}/2$ $\diamond$-neighbours in $B' \setminus X$. Therefore, as $\overline{\textbf{J8}}$ holds whenever $\mathcal{D} = \mathcal{H}$, and any edges in H'_i between $A' \setminus X$ and $B' \setminus X$ lie in $H_i \subset K_3$, we have $|A' \setminus X| \leq |B'|/100(\log n)^{1/3} \leq |A'|/2$. Thus, $|A' \cap X| \geq |A'|/2$.

Now, take $A'' \subset A' \cap X$ with $|A''| \geq |A'|/4$ so that all the $\diamond$-edges of $z_v \in A''$ were added under the same step **O1**–**O4** (noting that only two steps are used for each possible value of $\diamond$). If these edges were all added under **O1**, then let $A_0 = \{x_v : z_v \in A''\}$, and observe that every vertex in A_0 has at least $(\log \bar{n})^{2/3} - 2 \geq (\log n)^{2/3}/2$ in-neighbours in H_i in $B' \setminus X$. Therefore, again as $\overline{\textbf{J8}}$ holds whenever $\mathcal{D} = \mathcal{H}$, we have $|B'| \geq |B' \setminus X| \geq 100|A' \cap X|(\log n)^{1/3} > |A'|(\log n)^{1/3}$, a contradiction. A similar contradiction is reached if all these edges are added at **O2**, at **O3**, or at **O4**. Thus, no such sets A' and B' exist, so that **A3** holds for $D = H'_i$. Therefore, H'_i is a pseudorandom digraph with exceptional set X.

Modify C and P. For each $v \in Y$, recall that the labelled vertices d_v, b_v, a_v, c_v, e_v occur consecutively in this order on C. Note that, by **N2**, all these labelled vertices are distinct. Let C' be the cycle formed by, for each $v \in Y$, deleting the vertices a_v, b_v, c_v and adding the new vertex f_v along with the edges $d_v f_v$ and $f_v e_v$. Note these new edges are the same whether $d_v b_v$ or $b_v d_v$ is an edge of C, and whether $c_v e_v$ or $e_v c_v$ is an edge of C. Let P' be the path P with these same modifications carried out, so that $P' \subset C'$.

Modify $\bar{D}$. We modify $\bar{D}$ similarly to our modification for H_i. For each $v \in Y$, we wish to delete v, x_v and y_v from $\bar{D}$ and replace them with z_v (as created for H'_i), and add an in-edge from $w \in [n] \setminus Y'$ to z_v if there is an appropriate edge between w and x_v in $\bar{D}$ (matching the edge between d_v and b_v) and an out-edge if there is an appropriate edge between y_v and w in $\bar{D}$ (matching the edge between c_v and e_v). This will result in there being an edge from w to x_v with probability p, independently of all other edges in $\bar{D}$, and, similarly, an edge from x_v to w independently at random with probability p.

More precisely, let $\bar{D}'$ be the random graph formed by deleting $Y' = \{v, x_v, y_v : v \in Y\}$ from $\bar{D}$, and adding the vertices z_v, $v \in Y$, and the following edges for each $v \in Y$ and $w \in [n] \setminus Y'$.

P1 wz_v if $d_v b_v \in E(C)$ and $wx_v \in E(\bar{D})$

P2 wz_v if $b_v d_v \in E(C)$ and $x_v w \in E(\bar{D})$

P3 $z_v w$ if $c_v e_v \in E(C)$ and $y_v w \in E(\bar{D})$

P4 $z_v w$ if $e_v c_v \in E(C)$ and $wy_v \in E(\bar{D})$

Finally, for each distinct $v, v' \in Y$, add the edge $z_v z_{v'}$ independently at random with probability p. Note that the distribution of $\bar{D}'$ is the same as the binomial random digraph with vertex set $V(H'_i)$ and edge probability p.

Apply Theorem 2.3. Define $f : X \to V(P')$ by letting $f(z_v) = f_v$ for each $v \in Y$. Suppose $H'_i \cup \bar{D}'$ contains some C_0 which is a copy of C' in which f_v is copied to z_v for each $v \in Y$. We will show that, then, $H_i \cup \bar{D}$ contains a copy of C.

For each $v \in Y$, let α_v and β_v be the copies of d_v and e_v on C_0, respectively. Let P_v be the path $C[\{d_v, b_v, a_v, c_v, e_v\}]$. We will show that $\phi_v : P_v \to H_i \cup \bar{D}$ defined by $\phi_v(d_v) = \alpha_v$, $\phi_v(b_v) = x_v$, $\phi_v(a_v) = v$, $\phi_v(c_v) = y_v$ and $\phi_v(e_v) = \beta_v$ is an embedding of P_v into $H_i \cup \bar{D}$, for each $v \in Y$.

By the choice of C', we have $d_v f_v, f_v e_v \in E(C')$, and hence $\alpha_v z_v, z_v \beta_v \in E(H'_i \cup \bar{D}')$. If $d_v b_v \in E(C)$ and $\alpha_v z_v \in E(H'_i)$, then, as $\alpha_v z_v$ was added to H'_i at **O1**, we have $\alpha_v x_v \in E(H_i)$. If $d_v b_v \in E(C)$ and $\alpha_v z_v \in E(\bar{D}')$, then, by the choice of edges at **P1**, we have $\alpha_v x_v \in E(\bar{D})$. Thus, if $d_v b_v \in E(C)$, then $\alpha_v x_v \in E(H_i) \cup E(\bar{D})$. Similarly, considering **O2** and **P2**, if $b_v d_v \in E(C)$, then $x_v \alpha_v \in E(H_i) \cup E(\bar{D})$. Therefore, ϕ_v restricted to $\{d_v, b_v\}$ is an embedding of $P_v[\{d_v, b_v\}]$ into $H_i \cup \bar{D}$. Similarly, from **O3**, **O4**, **P3**, and **P4**, we have that ϕ_v restricted to $\{c_v, e_v\}$ is an embedding of $P_v[\{c_v, e_v\}]$ into $H_i \cup \bar{D}$. Finally, by the labelling after **N1**–**N2**, we have that ϕ_v restricted to $\{b_v, a_v, c_v\}$ is an embedding of $P_v[\{b_v, a_v, c_v\}]$ into $H_i \cup \bar{D}$. Thus, $\phi_v : P_v \to H_i \cup \bar{D}$ is an embedding of P_v into $H_i \cup \bar{D}$, for each $v \in Y$. As C' was formed by replacing $C[d_v, b_v, a_v, c_v, e_v]$ by $d_f \to f_v \to e_v$, for each $v \in Y$, if we take each path $\alpha_v z_v \beta_v$, $v \in Y$, on C_0, and replace it with $\phi(P_v)$, we get a copy of C in $H_i \cup \bar{D}$.

Finally, by Theorem 2.3, with probability at least $1-2\exp(-2\bar{n})$, $H_i' \cup \bar{D}'$ contains a copy of C' in which f_v is copied to z_v for each $v \in Y$. As, whenever this happens, $H_i \cup \bar{D}$ contains a copy of C, we have

$$\mathbb{P}(C \subsetneq H_i \cup \bar{D}) \geq 1 - \exp(-2\bar{n}) \geq 1 - \exp(-n).$$

This finishes the proof of the claim, and thus the theorem. □ □

4.4 Proof of Theorem 1.1

We now show how Theorem 1.1 follows from Theorem 2.4.

Proof of Theorem 1.1 from Theorem 2.4. Note first that, by **J11** in Lemma 4.2, we have, with high probability, that $m_1 \geq i_2 = 3n\log n/4$. Furthermore, D_{m_1+1} has no vertices with in- or out-degree 0, while D_{m_1} has exactly one such vertex. By Theorem 2.4, then, for both *(i)* and *(ii)* of Theorem 1.1 it is sufficient to show that, with high probability, there are no vertices $v \in V(D_{i_2})$ with $d^+_{D_{i_2}}(v), d^-_{D_{i_2}}(v) \leq 1$.

Let $p_1 = (i_2 - n)/n(n-1)$ and $D = D(n, p_1)$. Similarly as for (11), by Lemma 3.8, we have that $\mathbb{P}(e(D) \leq i_2) = 1 - o(1)$. Therefore, it is sufficient to show that, with high probability, D has no vertices $v \in V(D)$ with $d^+_D(v), d^-_D(v) \leq 1$. The probability that such a vertex does exist is at most

$$n\sum_{k=0}^{2}\binom{2n-2}{k}p_1^k(1-p_1)^{2(n-1)-k} \leq 3n(2p_1n)^2\exp(-p_1(2n-4)) = o(1),$$

as required.

For *(iii)* in Theorem 1.1, by Theorem 2.4 it is sufficient to show that, with high probability, D_{m_0} has at most $n^{1/2}\log^2 n$ vertices with in-degree 0 or out-degree 0, and at most $\log^5 n$ vertices $v \in D_{m_0}$ with $d^+_{D_{m_0}}(v) = d^-_{D_{m_0}}(v) = 1$.

From **J10** in Lemma 4.3, we have, with high probability, that $m_0 \geq i_1 = (n\log n)/2 - n\log\log n$. Let $p_2 = (i_1 - n)/n(n-1)$ and $\hat{D} = D(n, p_2)$. Similarly as for (11), by Lemma 3.8, we have that $\mathbb{P}(e(\hat{D}) \leq i_1) = 1 - o(1)$. Therefore, it is sufficient to show that, with high probability, D_2 has at most $n^{1/2}\log^2 n$ vertices with in-degree 0 or out-degree 0, and at most $\log^5 n$ vertices $v \in V(D_2)$ with $d^+_{\hat{D}}(v), d^-_{\hat{D}}(v) \leq 1$.

Let X_1 be the number of vertices with in-degree 0 or out-degree 0 in $\hat{D}$. Then,

$$\mathbb{E}X_1 \leq 2n(1-p_2)^{n-1} \leq 2n\exp(-p_2(n-1)) = O(n^{1/2}\log n).$$

Therefore, with high probability, $X_1 \leq n^{1/2}\log^2 n$.

Let X_2 be the number of vertices $v \in V(\hat{D})$ with $d^+_{\hat{D}}(v), d^-_{\hat{D}}(v) \leq 1$. Then,

$$\mathbb{E}X_2 \leq n\sum_{k=0}^{2}\binom{2n-2}{k}p_1^k(1-p_1)^{2(n-1)-k} \leq 3n(2p_1n)^2\exp(-p_2(2n-4)) = O(\log^4 n).$$

Therefore, with high probability, $X_2 \leq \log^5 n$. This completes the proof of *(iii)* in Theorem 1.1. □

4.5 Proof of Theorem 1.3

Finally, we deduce Theorem 1.3 from Theorem 2.4.

Proof of Theorem 1.3 from Theorem 2.4. Let $\mathcal{C}_n$ be the set of all n-vertex oriented cycles whose underlying undirected cycle is the canonical cycle with vertex set $[n]$. Recall that, for each $C \in \mathcal{C}_n$, $\lambda(C)$ is the number of vertices of C with in- or out-degree 0, and $p_C = \max\{\log n, 2(\log n - \lambda(C))\}/2n$ if $\lambda(C) > 0$, and $p_C = \log n/n$ otherwise.

Let $\varepsilon > 0$ be small and fixed and $p = p(n)$. Note that, for Theorem 1.3, we can assume that $(\min_{C\in\mathcal{C}_n} p_C)/(1-\varepsilon) \leq p \leq (\max_{C\in\mathcal{C}_n} p_C)/(1+\varepsilon)$, and thus that $\log n/2(1-\varepsilon)n \leq p \leq \log n/(1+\varepsilon)n$.

Now, $(1+\varepsilon)p \geq (1+\varepsilon)\log n/2(1-\varepsilon)n > \log n/2n$, so if we have $C \in \mathcal{C}_n$ and $p_C \geq (1+\varepsilon)p$, then

$$\frac{\log n - \log \lambda(C)}{n} = p_C \geq (1+\varepsilon)p,$$

and hence C has at most $n\exp(-(1+\varepsilon)pn)$ vertices with in-degree 0. On the other hand, we will show that $D = D(n,p)$ is likely to have more than $n\exp(-(1+\varepsilon)pn)$ vertices with in-degree 0, and thus contain no such cycle. Note that, for each $v \in V(D)$, the probability that $d^-_D(v) = 0$ is $(1-p)^{n-1} \geq \exp(-(1+\varepsilon/2)pn)$. Furthermore, this is independent for each $v \in V(D)$. Thus, as $(1+\varepsilon)pn \leq \log n$ and the expected number of vertices with $d^-_D(v) = 0$ is at least $n \cdot \exp(-(1+\varepsilon/2)pn)$, by Lemma 3.8, with high probability there are more than $n\exp(-(1+\varepsilon)pn)$ vertices with degree 0 in D. Thus, with high probability, $D(n,p)$ contains no cycle $C \in \mathcal{C}_n$ with $p_C \geq (1+\varepsilon)p$.

Now, if $C \in \mathcal{C}_n$ has $p_C \leq (1-\varepsilon)p$, then

$$\frac{\log n - \log \lambda(C)}{n} \leq (1-\varepsilon)p,$$

and hence C has at least $n\exp(-(1-\varepsilon)pn)$ vertices with in-degree 0. On the other hand, the expected number of vertices in $D(n,p)$ with out-degree 0 or in-degree 0 is at most

$$2n(1-p)^{n-1} \leq 2n\exp(-p(n-1)) = o(n\exp(-(1-\varepsilon)pn)/\log^2 n).$$

Therefore, with high probability, $D(n,p)$ has at most $n\exp(-(1-\varepsilon)pn)/\log^2 n$ vertices with in- or out-degree 0. Furthermore, as $\log n/2(1-\varepsilon)n \leq p \leq \log n/(1+\varepsilon)n$, the probability that $D(n,p)$ contains a vertex with total in- and out-degree less than 3 is at most

$$n \cdot \sum_{i=0}^{2} \binom{2n-2}{i} p^i(1-p)^{2(n-1)-i} \leq 3n(2np)^2\exp(-p(2n-4)) = o(1).$$

Therefore, with high probability, each vertex in $D(n,p)$ has total in- and out-degree at least 3 (and hence, in particular, no vertices with in- and out-degree both 1).

Let $\mathcal{P}$ be the property of digraphs D such that, for all s, t and n, if D has n vertices, s of which have in-degree 0 or out-degree 0 and t of which have in-degree 1 and out-degree 1, and $d^+_D(v) + d^-_D(v) \geq 2$, then D contains a copy of every n-vertex cycle with at least $1 + (s-1)\log n$ changes in direction and at most $n - 1 - (t-1)\log n$ changes in direction. Thus, it is sufficient to complete the proof of the theorem to show that, with high probability, $D(n,p) \in \mathcal{P}$.

Let $\eta > 0$. By Theorem 2.4, there is some n_0 such that, for each $0 \leq M \leq n(n-1)$, if $D_{n,M}$ is a random digraph chosen uniformly from those with vertex set $[n]$ and M edges, then $\mathbb{P}(D_{n,M} \in \mathcal{P}) \geq 1 - \eta$. Then,

$$\mathbb{P}(D(n,p) \in \mathcal{P}) = \sum_{M=0}^{n(n-1)} \mathbb{P}(e(D(n,p)) = M) \cdot \mathbb{P}(D_{n,M} \in \mathcal{P}) \geq \sum_{M=0}^{n(n-1)} \mathbb{P}(e(D(n,p)) = M) \cdot (1-\eta) = (1-\eta).$$

Thus, as n_0 was chosen depending only on η, we have, with high probability, that $D(n,p)$ is in $\mathcal{P}$, as required. □

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