Ramsey numbers of bounded degree trees versus general graphs

Richard Montgomery* Matías Pavez-Signé† Jun Yan‡

Abstract

For every $k \geq 2$ and Δ, we prove that there exists a constant $C_{\Delta,k}$ such that the following holds. For every graph H with $\chi(H) = k$ and every tree with at least $C_{\Delta,k}|H|$ vertices and maximum degree at most Δ, the Ramsey number $R(T, H)$ is $(k-1)(|T|-1) + \sigma(H)$, where $\sigma(H)$ is the size of a smallest colour class across all proper k-colourings of H. This is tight up to the value of $C_{\Delta,k}$, and confirms a conjecture of Balla, Pokrovskiy, and Sudakov.

1 Introduction

Given two graphs G and H, the *Ramsey number* $R(G, H)$ is defined as the smallest integer N such that every red/blue colouring of the edges of the complete graph K_N contains either a red copy of G or a blue copy of H. The fundamental result of Ramsey [25] implies that $R(G, H)$ is well-defined for any pair of graphs G and H. The exact value of $R(G, H)$ is known for very few pairs of graphs (G, H), and in general it is difficult even to give good bounds on $R(G, H)$. Of the few exact Ramsey numbers known, many share the same general extremal lower bound construction given below. The area of *Ramsey goodness* studies the graphs G and H for which this lower bound is tight.

Erdős showed in 1947 that the Ramsey number of an n-vertex path P_n and a complete m-vertex graph K_m is $R(P_n, K_m) = (m-1)(n-1)+1$. Here, the lower bound construction is the disjoint union of $m-1$ red $(n-1)$-vertex cliques with all possible edges in blue added between them. As Chvátal and Harary [9] observed, this construction contains no connected red n-vertex subgraph and no blue subgraph with chromatic number at least m; if G is connected, we therefore have $R(G, H) \geq (\chi(H)-1)(|G|-1)+1$. Let $\sigma(H)$ denote the size of a smallest colour class across all proper $\chi(H)$-colourings of H. If $2 \leq \sigma(H) \leq |G|$, then this construction can be improved by adding further a red clique of order $\sigma(H) - 1$ connected to the rest of the construction with blue edges. This observation, made by Burr [6], implies that for every connected graph G with $|G| \geq \sigma(H)$ we have

$$R(G, H) \geq (|G| - 1)(\chi(H) - 1) + \sigma(H). \tag{1.1}$$

As coined by Burr and Erdős [7] in 1983, we say that a graph G is *H-good* if (1.1) holds with equality, that is, if $R(G, H) = (|G| - 1)(\chi(H) - 1) + \sigma(H)$. A family of graphs $\mathcal{G}$ is said to be *H-good* if there is some n_0 such that every graph $G \in \mathcal{G}$ is H-good if $|G| \geq n_0$. Chvátal [8] had already shown in 1977 that the family of trees is K_r-good for all r. Inspired by this, Burr and Erdős sought to determine which families $\mathcal{G}$ are K_r-good for all r. They showed that, for each fixed b, the family of graphs with *bandwith* at most b is K_r-good for all r, and raised a series of questions concerning the K_r-goodness of various natural families [7].

In 1985, Erdős, Faudree, Rousseau, and Schelp [11] showed that the family of bounded degree trees is H-good for any graph H, where some (possibly much weaker) restriction on the maximum degree is needed

*Mathematics Institute, University of Warwick, Coventry CV4 7AL, UK. richard.montgomery@warwick.ac.uk
†Centro de Modelamiento Matemático (CNRS IRL2807), Universidad de Chile, Santiago, Chile. mpavez@dim.uchile.cl
‡Mathematics Institute, University of Warwick, Coventry CV4 7AL, UK. jun.yan@warwick.ac.uk.

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(see [5]). All but one of the Burr-Erdős questions raised in [7] were solved by Nikiforov and Rousseau [22] in 2009, mostly as a result of their proof of the K_r-goodness for each r of any family of graphs with constant maximum degree which have a 'small' set of vertices whose removal divides the graph into small linear-sized components (see [22] for a more precise statement). The last question in [7] was solved by Fiz Pontiveros, Griffiths, Morris, Saxton, and Skokan [12] in 2014, who proved that the n-dimensional hypercube Q_n is H-good if n is sufficiently large. Beyond these questions, Allen, Brightwell, and Skokan [1] showed in 2013 that the family of graphs G with constant maximum degree and bandwith $o(|G|)$ is H-good for every graph H.

More recently, focus has been turned on quantitative considerations of the Ramsey goodness problem. That is, when $\mathcal{G}$ is an H-good family, how large does n_0 need to be before every $G \in \mathcal{G}$ is H-good if $|G| \geq n_0$. As their result quoted above shows that for any graph H, the path P_n and the n-vertex cycle C_n are H-good if n is sufficiently large depending on H, Allen, Brightwell, and Skokan [1] conjectured that $n \geq \chi(H)|H|$ should be sufficient. In the case of paths, this was proved in a strong form when $\chi(H) \geq 4$ by Pokrovskiy and Sudakov [23], who showed that, if $n \geq 4|H|$, then P_n is H-good. For cycles, Pokrovskiy and Sudakov [24] later showed that C_n is H-good as long as $n \geq 10^{60}|H|$ and $\sigma(H) \geq \chi(H)^{22}$. This result was recently improved by Haslegrave, Hyde, Kim, and Liu [16], who showed that, for some universal $C > 0$, $n \geq C|H| \log^4 \chi(H)$ is sufficient for the cycle C_n to be H-good, which is optimal up to the logarithmic factor of $\chi(H)$ and confirms the conjecture of Allen, Brightwell, and Skokan for graphs with large enough chromatic number.

Though not mentioned explicitly, quantitative bounds can be taken more generally for the Ramsey goodness of bounded degree trees from the proof of Erdős, Faudree, Rousseau, and Schelp [11] mentioned above. That is, the proof can be used to show that, for each k and Δ, there is some $C_{\Delta,k}$ such that for each graph H with $\chi(H) = k$ and each $n \geq C_{\Delta,k}|H|^4$, every n-vertex tree T with maximum degree at most Δ is H-good. This was improved by Balla, Pokrovskiy, and Sudakov [2] in 2018, who proved that $n \geq C_{\Delta,k}|H| \log^4 |H|$ suffices, for some appropriate $C_{\Delta,k}$. Furthermore, they showed the factor $\log^4 |H|$ is not needed when T has $\Omega(n)$ leaves, and conjectured this should be true for every tree, that is, that $n \geq C_{\Delta,k}|H|$ should suffice for T to be H-good. The purpose of this paper is to confirm this conjecture, as follows.

Theorem 1.1. *For all Δ and k, there exists a constant $C_{\Delta,k}$ such that the following holds. Given a graph H with $\chi(H) = k$ and $n \geq C_{\Delta,k}|H|$, every n-vertex tree T with maximum degree at most Δ is H-good. In other words, $R(T, H) = (k-1)(n-1) + \sigma(H)$.*

Theorem 1.1 is tight up to the value of $C_{\Delta,k}$, as it is easy to see that $R(G, H) \geq |H|$ for any non-empty graphs G and H, so G is not H-good if $|G| < |H|/k$, and hence $C_{\Delta,k} = \Omega(1/k)$. However, without much more difficulty, it can be seen that we must have $C_{\Delta,k} \geq \Delta/100k \log \Delta$ for sufficiently large Δ, by taking m to be sufficiently large, letting H be the k-partite complete graph with m vertices in each class, and letting T be any tree with maximum degree Δ on $n = \Delta m/100 \log \Delta$ vertices. To see this, let $N = \Delta m/10 \log \Delta$, and take a blue complete $\lfloor k/3 \rfloor$-partite graph with N vertices in each class, and colour the edges within each class red/blue so that the maximum red degree is at most $\Delta - 1$ and there is no blue copy of $K_{m,m}$. This colouring is possible using the probabilistic method, more precisely, colouring edges of K_{2N} red independently at random with probability $\Delta/4N$ and removing vertices with at least Δ red neighbouring edges. Call the final red/blue coloured graph G. If there is a blue copy of H in G, then at least $3m$ vertices from this copy are within one class of N vertices in the original partition of G, and hence the colouring within this class must contain a blue copy of $K_{m,m}$, a contradiction. As G has maximum red degree at most $\Delta - 1$, it contains no red copy of T, and therefore $R(T, H) > \lfloor k/3 \rfloor N > kn$. Thus, T is not H-good, so we must have $C_{\Delta,k} \geq |T|/|H| \geq \Delta/100k \log \Delta$. We have not optimised the value of $C_{\Delta,k}$ in Theorem 1.1 as our proof would give an upper bound on $C_{\Delta,k}$ that is very far from this $\Delta/100k \log \Delta$ lower bound.

2 Preliminaries

After covering the basic notation that we use, we will give a detailed sketch of the proof of Theorem 2.1 in Section 2.2 before outlining the rest of the paper in Section 2.3.

2.1 Notation

A graph G has vertex set $V(G)$ and edge set $E(G)$, and we write $|G| = |V(G)|$ and $e(G) = |E(G)|$. A vertex $v \in V(G)$ has neighbourhood $N(v)$ and degree $d(v) = |N(v)|$. Given a vertex set $S \subset V(G)$, the set of neighbours of S (including those in S) is $N'(S) = \cup_{s\in S}N(s)$ and its external neighbourhood is $N(S) = \cup_{s\in S}N(s) \setminus S$. For a vertex $v \in V(G)$ and sets $U, S \subset V(G)$, we write $d(v, U) = |N(v) \cap U|$ and $N(S, U) = N(S) \cap U$. Given a subset $S \subset V(G)$, $G[S]$ is the subgraph of G induced on S, with vertex set S and every edge of G contained within S. We write $G - S$ for the graph $G[V(G) \setminus S]$. For a subset $X \subset V(G)$, $I(X)$ denotes the subgraph of G with vertex set X and no edges. The complement of G, denoted G^c, is the graph with $V(G^c) = V(G)$ and such that xy forms an edge in G^c if and only if xy is not an edge in G.

Given two subgraphs $H_1, H_2 \subset G$, $H_1 \cup H_2$ is the subgraph with vertex set $V(H_1) \cup V(H_2)$ and edge set $E(H_1) \cup E(H_2)$. If $H \subset G$ is a subgraph and $e = xy \in E(G)$, then $H + e$ denotes the subgraph with vertex set $V(H) \cup \{x, y\}$ and edge set $E(H) \cup \{xy\}$. Given $x, y \in V(G)$, an x, y-path is a path with endvertices x and y. A path P with ℓ vertices has length $\ell - 1$. We use $H + P$ to denote $H \cup P$, and $H - P$ to denote the graph resulting from removing the edges of P from H and any internal vertex of P which is now isolated. For subsets $S_1, S_2 \subset V(G)$, an S_1, S_2-path is a path with one endpoint in S_1 and the other endpoint in S_2.

For a positive integer $n \in \mathbb{N}$, we write $[n] = \{1, \dots, n\}$ and $[n]_0 = [n] \cup \{0\}$. We will use the standard hierarchy notation, that is, for $a, b \in (0, 1]$, we will write $a \ll b$ to mean that there exists a non-decreasing function $f : (0, 1] \to (0, 1]$ such that if $a \leq f(b)$ then the rest of the proof holds with a and b. Hierarchies with more constants are defined in a similar way and are to be read from the right to the left. For simplicity we will ignore floor and ceiling signs whenever this does not affect the argument.

2.2 Outline of the proof of Theorem 2.1

Let H have chromatic number k such that $|H| \leq \mu n$. Let $m = \sigma(H)$. For convenience, if not elegance, sake, we will use $K^{k-1}_{\mu n} \times K^c_m$ to denote the complete k-partite graph whose first $k - 1$ classes have size μn and whose last class has size m. Thus, if a graph G contains a copy of $K^{k-1}_{\mu n} \times K^c_m$, then it contains a copy of H, so that, to prove Theorem 1.1 it is enough to prove the following (where we also take $\mu_{\Delta,k} = 1/C_{\Delta,k}$).

Theorem 2.1. *For all Δ and k, there exists a constant $\mu_{\Delta,k}$ such that the following holds for each $m \leq \mu_{\Delta,k}n$. Every tree T with n vertices and maximum degree at most Δ satisfies $R(T, K^{k-1}_{\mu_{\Delta,k}n} \times K^c_m) = (k - 1)(n - 1) + m$.*

We will prove Theorem 2.1 by induction on k, and therefore the proof falls into two parts, the base case ($k = 2$) and the inductive step. This follows the work of Balla, Pokrovskiy, and Sudakov [2], who proved Theorem 2.1 under a stronger assumption equivalent to replacing $\mu_{\Delta,k}n$ with $\mu_{\Delta,k}n/\log^4 n$ throughout. The critical case for making our improvement is the case $k = 2$, and we concentrate on this in this sketch, before outlining briefly how the case for $k \geq 3$ follows by induction.

For the case $k = 2$, it is sufficient to prove the following for some $\mu = \mu(\Delta)$. That, for every n-vertex tree T with maximum degree Δ and all $m \leq \mu n$, the following holds.

A If G has $n + m - 1$ vertices, and the complement of G is $K_{m,\mu n}$-free, then G contains a copy of T.

An $(n + m - 1)$-vertex graph G whose complement is $K_{m,\mu n}$-free satisfies the natural expansion condition that, for each set U of m vertices, $|N_G(U)| \geq |G| - |U| - \mu n \geq (1 - 2\mu)n$. Beginning with work by Friedman and Pippenger [13], through a key development by Haxell [17], trees can be embedded into graphs satisfying expansion conditions (see, for example, Corollary 2.19). A typical such application of these methods would give the following.

B If every set U of m vertices in a graph G satisfies $|N(U)| \geq |G| - 2\mu n$, then a tree T with maximum degree Δ can be embedded in G if G has at least $20\Delta \cdot \max\{\mu n, m\}$ more vertices than T.

Our trouble, then, is the gulf between **A** and **B**: we only have $m - 1$ more vertices in G than T in **A**, far from the $20\Delta \cdot \mu n$ spare vertices required to use **B**, particularly as m can be any value between 1 and μn. Critically, we use the following observation for our graph G whose complement is $K_{m,\mu n}$-free. That, for some large constant K, if we take a set $V \subset V(G)$ of Km vertices at random, then, any set $U \subset V(G)$ of m vertices has at most μn non-neighbours in G, so can expect to have at most $2\mu Km$ non-neighbours in V. With a little care, then, we can show, with high probability, that every set $U \subset V$ with $|U| = m$ satisfies $|N_{G[V]}(U)| \geq (1 - 3\mu)|V|$. Then, in $G[V]$, we can apply **B** to embed a tree with maximum degree Δ using only $O(\Delta \cdot \max\{\mu|V|, m\}) = O(\Delta \cdot \max\{\mu Km, m\})$ spare vertices. When we embed using $m - 1$ spare vertices (as at **A**), this is still not enough, but for small m we are much closer, needing $O(m)$ spare vertices instead of $O(n)$ spare vertices. With more analysis, and choosing the constants involved carefully, we can show that, for some small λ, $G[V]$ here is likely to be $K_{\lambda m, \lambda m}$-free, which will allow us to use **B** to embed a tree with maximum degree Δ into $G[V]$ while using only $m - 1$ spare vertices.

This key idea for finishing the embedding leads to the following strategy. We take a nested sequence of random subsets $V(G) = U_0 \supset U_1 \supset \ldots \supset U_\ell$ with geometrically-decreasing size, before embedding T in ℓ stages, at each stage $i \in [\ell]$ using all unused vertices in $U_{i-1} \setminus U_i$ and as few vertices in U_i as possible, until we finish by embedding the last part of T within $G[U_\ell]$. This approach is inspired by the 'cover down methods' in the iterative absorption techniques used by the first author with Glock, Kühn, Lo, and Osthus [15] (developing earlier work by Barber, Kühn, Lo, and Osthus [3]), and similarly we also call our nested sequence of sets a *vortex*.

Between this sketch and our implementation, there are several complications. We cannot take a simple vortex and must run a cleaning process on an initial random vortex. The 'vortex' embedding method also requires m to be at least some large constant; when it is not we must use a separate (though much easier) embedding. Finally, before doing anything, we need to efficiently gain an expansion property for small sets (i.e., those with fewer than m vertices). As in previous work (in particular, [2]), we can remove a small set of vertices (effectively some small maximal set which does not expand) to gain this property, but removing these vertices gives us fewer 'spare vertices' to work with than the $m - 1$ we need. If we end up with $m' - 1$ spare vertices, then it will not be hard to show that sets with size m' expand significantly, but this change in the value of m needs to be dealt with carefully, which we do in Section 2.4.

Finally, we note that, while some of the tree embeddings we use are quite intricate, there exist very good methods to carry them out in a flexible fashion using expansion properties. More specifically, we use an 'extendability' method for embedding trees that combines the inductive embedding of trees in expanding graphs by Haxell [17] (developing work by Friedman and Pippenger [13]) with a 'roll-back' idea by Johannsen (see [10]), as used by Glebov, Johannsen, and Krivelevich [14], and Draganic, Krivelevich, and Nenadov [10]. For embedding part of the tree into our vortex while making sure we cover all of the unused vertices (as discussed above) we use results of the first author from [21], which we develop into the form we require in Section 4.

Case $k \geq 3$. Having proved the $k = 2$ case, we will then proceed by induction on k. To prove Theorem 2.1 for k, we will have a graph G with $(k-1)(n-1) + m$ vertices and an n-vertex tree T with maximum degree Δ, and $m \leq \mu n$ for some small $\mu > 0$, and look to show that either G contains a copy of T or G^c contains a copy of $K_{\mu n}^{k-1} \times K_m^c$. We proceed differently according to three cases, **a)**–**c)**, the proof of which are carried out in Sections 6.1–6.3 where a more detailed proof sketch for each case can be found.

a) In this case, T has linearly (in n) many leaves, and we follow a method of Balla, Pokrovskiy, and Sudakov [2].

b) In this case, G is not well connected in the sense that there is a small set V_0 such that $G - V_0$ has two large disjoint parts. We can assume (using induction) that the subgraphs of G induced on these large parts both satisfy an expansion condition. If there is a vertex with Δ neighbours in each part then we can embed part of the tree T in each part, connected together appropriately by this vertex. Where there is no such vertex, we partition $V(G)$ into two large sets which have few edges between them in G. If G has no copy of T, then we can apply induction to both subgraphs induced by the sets of this partition to find complete partite graphs in their complement, which we combine before removing some vertices in the larger classes in order to find our desired complete partite graph in G^c.

c) In the final case, G is well-connected and T has few leaves. The embedding we use here is the most complicated one in the inductive step. Effectively, though, we use expansion conditions to embed T with many bare paths (i.e., paths which no branching vertices in T) removed, which exist as T has few leaves, before using a results on the Ramsey goodness of paths to find paths that we then connect into the embedding of T using the connectivity property. This sketch serves more to motivate our eventual embedding, which cannot follow this sketch closely, but will be discussed in full in Section 6.3.

2.3 Organisation of the paper

In the rest of this section, we will first show, in Section 2.4 that the case $k = 2$ of Theorem 2.1 can be divided into 2 subcases, covered by Theorem 2.4 and Theorem 2.5, where the latter represents the critical case sketched in Section 2.2. We then find our tree decompositions (Section 2.5), give some basic results involving expansion properties (Section 2.6), and cover the tree embedding tools we will use (in Section 2.7). In Section 3, we prove Theorem 2.4. In Section 4, we show how to embed trees while covering a vertex set, to be applied at each stage when embedding the tree in the critical case. This then allows us to prove the result in this critical case, Theorem 2.5, in Section 5. Finally, in Section 6, we use induction on k to prove Theorem 2.1 from the $k = 2$ case.

2.4 Subcases when $k = 2$

As discussed in Section 2.2, the case when $k = 2$ in Theorem 2.1 is equivalent to the following result, which for convenience we state separately.

Theorem 2.2. *Let $\mu \ll 1/\Delta$ and $1 \leq m \leq \mu n$, and let T be an n-vertex tree with $\Delta(T) \leq \Delta$. Then, $R(T, K_{m,\mu n}) = n + m - 1$.*

We will separate Theorem 2.2 into two subcases, but first need the following definition for the main forms of expansion used throughout this paper.

Definition 2.3. Let $m, m' \in \mathbb{N}$, $d > 0$ and let G be a graph.

i) We say that G is a (d, m)-*expander* if $|N(U)| \geq d|U|$ for all subsets $U \subset V(G)$ with $|U| \leq m$.

ii) We say that G is (m, m')-*joined* if every pair of disjoint sets A and B, with $|A| = m$ and $|B| = m'$, has an edge between them in G. When $m = m'$, we simply say the graph is m-*joined*.

Note that a graph is (m, m')-joined exactly when its complement is $K_{m,m'}$-free. Using these definitions, we can now state two results, Theorems 2.4 and 2.5, before proving that Theorem 2.2 follows from them using an additional result, Proposition 2.6. Theorem 2.4 covers Theorem 2.2 when m is at most some constant depending on μ, while Theorem 2.5, roughly speaking, will cover Theorem 2.2 when m is larger and some stronger expansion condition holds. For convenience in their proofs, we adjust the parameters so that they are applied to trees with $n - m + 1$ vertices in an n-vertex graph (instead of n and $n + m - 1$, respectively).

Theorem 2.4. *Let $\mu \ll 1/\Delta, 1/m$ with $m \leq \mu n$, and let G be an n-vertex $(m, \mu n)$-joined graph.*
Then, G contains a copy of every $(n - m + 1)$-vertex tree T with $\Delta(T) \leq \Delta$.

Theorem 2.5. *Let $1/D, 1/m, \mu \ll 1/\Delta$ and $m \leq \mu n$, and let G be an n-vertex $(m, \mu n)$-joined graph which is a (D, m)-expander.*
Then, G contains a copy of every $(n - m + 1)$-vertex tree T with $\Delta(T) \leq \Delta$.

Before showing these two results imply Theorem 2.2, we need to find an expander in the sense of Definition 2.3i) in our graphs covered by this theorem, which we do in the following proposition.

Proposition 2.6. *Let $n_0, m \in \mathbb{N}$ and $d > 0$, and let G be an (m, n_0)-joined graph with $|G| \geq n_0 + (2d+2)m$. Then, there exists some $W \subset V(G)$ such that $|W| < m$ and, setting $m' = m - |W|$, $G - W$ is an $(m', n_0 + dm)$-joined (d, m)-expander.*

Proof. Let $W \subset V(G)$ be a maximal subset subject to $|W| < 2m$ and $|N_G(W)| \leq d|W|$, possible as $W = \emptyset$ satisfies these conditions. Note that if $|W| \geq m$, then, as G is (m, n_0)-joined, $|V(G) \setminus (W \cup N_G(W)| < n_0$, so that $|N_G(W)| \geq |G| - |W| - n_0 \geq 2dm > d|W|$, a contradiction. Thus, we have $|W| < m$. Let $m' = m - |W| \geq 1$.

Now, if $U \subset V(G) \setminus W$ with $|U| \leq m$ and $U \neq \emptyset$, then, as $|W| < m$, we must have that $|U \cup W| < 2m$ and $|U \cup W| > |W|$. To avoid contradicting the choice of W, we must have $|N_G(U \cup W)| \geq d|U \cup W|$, and hence $|N_{G-W}(U)| \geq d|U \cup W| - d|W| \geq d|U|$. Thus, $G - W$ is an (d, m)-expander.

Furthermore, if $X, Y \subset V(G - W)$ are disjoint sets with $|X| = m'$, $|Y| = n_0 + dm$, and $e_{G'}(X, Y) = 0$, then $e_G(X \cup W, Y \setminus N_G(W)) = 0$, $|X \cup W| = m$ and $|Y \setminus N_G(W)| \geq n_0 + dm - d|W| \geq n_0$, which contradicts that G is (m, n_0)-joined. Thus, $G - W$ is $(m', n_0 + dm)$-joined. □

We can now deduce Theorem 2.2 from Theorems 2.4 and 2.5 using Proposition 2.6, as follows.

Proof of Theorem 2.2. Let $\mu \ll 1/\Delta$ and $1 \leq m \leq \mu n$, let T be an n-vertex tree with $\Delta(T) \leq \Delta$, and let G be a graph on $n+m-1$ vertices such that G^c does not contain a copy of $K_{m,\mu n}$. To prove Theorem 2.2, then, we need to show that G contains a copy of T. Using that $\mu \ll 1/\Delta$, let $\bar{\mu}$ and $\bar{m}$ satisfy $\mu \ll \bar{\mu} \ll 1/\bar{m} \ll 1/\Delta$ and let $D = \bar{\mu}n/100m \geq \bar{\mu}/100\mu$ (as $m \leq \mu n$), so that $1/D \ll 1/\Delta$.

We have $n \geq \bar{\mu}n/2 + (2D+2)m$, and that G is $(m, \bar{\mu}n/2)$-joined as its complement is $K_{m,\mu n}$-free and $\mu \ll \bar{\mu}$. Thus, using Proposition 2.6 and $Dm \leq \bar{\mu}n/2$, we can find a set $W \subset V(G)$ with $|W| < m$ such that, setting $m' = m - |W|$ and $G' = G - W$, G' is an $(m', \bar{\mu}n)$-joined (D, m)-expander. Note that $|G'| = n + m' - 1$ and $m' \leq m \leq \mu n \leq \bar{\mu}n$.

If we have $m' < \bar{m}$, then, as $\bar{\mu} \ll 1/\bar{m}, 1/\Delta$ and G' is an $(m', \bar{\mu}n)$-joined graph with $n + m' - 1$ vertices, G' contains a copy of T by Theorem 2.4. If $m' \geq \bar{m}$, then since G' is a (D, m)-expander and $m' \leq m$, it is also a (D, m')-expander. Then, as $1/\bar{m}, 1/D, \bar{\mu} \ll 1/\Delta$, $m' \geq \bar{m}$, and G' is $(m', \bar{\mu}n)$-joined, G' contains a copy of T by Theorem 2.5, as required. □

2.5 Structural decompositions of trees

To show our cases cover all trees, we will need the following useful result which states that, if a tree has few leaves, then it has many bare paths in compensation.

Lemma 2.7 ([20, Lemma 2.1])**.** *Let $k, \ell, n \in \mathbb{N}$ and let T be a tree with n vertices and at most ℓ leaves. Then T contains at least $\frac{n}{k+1} - 2\ell + 2$ vertex disjoint bare paths, each of length k.*

We now give our main tree decomposition, decomposing the edges of a bounded degree tree T into subtrees $T_1, \ldots, T_\ell$ such that T_i and T_{i+1} intersect only on a leaf of T_i, as in the following definition.

Definition 2.8. Let $\ell \in \mathbb{N}$ and let $\mathbf{n} = (n_1, \ldots, n_\ell) \in \mathbb{N}^\ell$. An $\mathbf{n}$-decomposition of a tree T is a tuple $(T_1, \ldots, T_\ell)$ of edge-disjoint subtrees of T such that

i) $E(T) = E(T_1) \cup \ldots \cup E(T_\ell)$,

ii) $|T_1| = n_1$, and $|T_i| = n_i + 1$ for each $2 \leq i \leq \ell$, and

iii) for each $i \in [\ell - 1]$, T_i contains no vertices in $T_{i+2} \cup \ldots \cup T_\ell$, and $V(T_i) \cap V(T_{i+1})$ contains exactly one vertex, which is a leaf of T_i.

We want to find such a decomposition where the trees $T_1, \ldots, T_\ell$ decrease in size by approximately a constant factor each time. That is, an $\mathbf{n}$-decomposition where $\mathbf{n}$ is (γ_1, γ_2)-*descending*, as follows.

Definition 2.9. For $0 < \gamma_1 < \gamma_2 < 1$, we say that a tuple $\mathbf{n} = (n_1, \ldots, n_\ell) \in \mathbb{N}^\ell$ is (γ_1, γ_2)-*descending* if $\gamma_1 n_i \leq n_{i+1} \leq \gamma_2 n_i$ for every $i \in [\ell - 1]$.

We will find descending decompositions of trees, by iteratively using the following proposition.

Proposition 2.10. *Let $\gamma \in (0,1)$ and $n, \Delta \in \mathbb{N}$ satisfy $\gamma n \geq \Delta$ and $n \geq 2$. Let T be a tree with n vertices and $\Delta(T) \leq \Delta$, and let $t \in V(T)$. Then, there exists two subtrees $T_1, T_2 \subseteq T$ such that*

i) *$E(T) = E(T_1) \cup E(T_2)$,*

ii) *T_1 and T_2 share exactly one vertex v, which is a leaf of T_1,*

iii) *$t \in V(T) \setminus V(T_2)$, and*

iv) *$\gamma n \geq |T_2| \geq \gamma n/2\Delta$.*

Proof. View T as being rooted at t. Let v be a vertex furthest from t subject to the condition that the subtree T_2 of T rooted at v has size $|T_2| \geq \gamma n/2\Delta$. Note that this is possible as t has at most $\Delta(T) \leq \Delta$ neighbours, the subtree rooted at one of these must have size at least $(n-1)/\Delta \geq n/2\Delta$ and hence is a candidate for v as $\gamma < 1$. Moreover, this shows $v \neq t$. Let T_1 be subgraph of T induced by the vertex set $(V(T) \setminus V(T_2)) \cup \{v\}$. Note that T_1 is a subtree of T and v is leaf in T_1. It follows that T_1, T_2 satisfies conditions i), ii) and iii). For condition iv), let v' be any neighbour of v in T_2. To avoid contradicting the definition of v, the subtree rooted at v' must have size at most $\lfloor \gamma n/2\Delta \rfloor$. Since there are at most Δ such v', we have $|T_2| \leq 1 + \Delta \cdot \lfloor \gamma n/2\Delta \rfloor \leq \gamma n$, as required. □

We can now find our desired descending decompositions of large bounded degree trees, as follows.

Lemma 2.11. *Let $0 < \gamma < 1/2$ and let $n, N, \Delta \in \mathbb{N}$ satisfy $n - 1 > N \geq \Delta/\gamma$. Then for any n-vertex tree T with maximum degree at most Δ and any $t \in V(T)$, there exists a $(\frac{\gamma}{4\Delta}, 2\gamma)$-descending tuple $\mathbf{n} = (n_1, \ldots, n_\ell) \in \mathbb{N}^\ell$ with $\frac{\gamma}{3\Delta}N \leq n_\ell \leq N$ such that T has an $\mathbf{n}$-decomposition $(T_1, \ldots, T_\ell)$ with $t \in V(T_1)$.*

Proof. We construct the desired decomposition iteratively, beginning with $T_1' = T$, and after i iterations obtain a sequence of trees $(T_1, T_2, \ldots, T_{i-1}, T_i')$ forming a $(|T_1|, |T_2|-1, \ldots, |T_{i-1}|-1, |T_i'|-1)$-decomposition of T. For each $i \geq 1$, if $|T_i'| - 1 \leq N$, then stop iterating. Otherwise, let t_i be the unique common vertex of T_i' and T_{i-1} if $i \geq 2$ and let $t_1 = t$. Noting that $|T_i'| > N \geq \Delta/\gamma$, we can apply Proposition 2.10 to obtain subtrees T_i and T_{i+1}' of T_i' such that $E(T_i') = E(T_i) \cup E(T_{i+1}')$, T_i and T_{i+1}' share a unique vertex t_{i+1} which is a leaf of T_i, $t_i \in V(T_i) \setminus V(T_{i+1}')$, and $\gamma|T_i'| \geq |T_{i+1}'| \geq \gamma|T_i'|/2\Delta$. This implies that $(T_1, T_2, \ldots, T_i, T_{i+1}')$ is a $(|T_1|, |T_2|-1, \ldots, |T_i|-1, |T_{i+1}'|-1)$-decomposition of T.

As $\gamma < 1$, this process must end; suppose it ends with a sequence $(T_1, \ldots, T_{\ell-1}, T_\ell')$. Let $n_1 = |T_1|$, $n_i = |T_i| - 1$ for each $2 \leq i \leq \ell - 1$, $n_\ell = |T_\ell'| - 1$, and $\mathbf{n} = (n_1, \ldots, n_\ell)$, so that $(T_1, \ldots, T_{\ell-1}, T_\ell')$ is an $\mathbf{n}$-decomposition of T. Noting that $\ell \geq 2$ as $|T_1'| = n > N + 1$, and since the process stopped after ℓ and not $\ell - 1$ iterations, we have $n_\ell \leq N$ and $|T_{\ell-1}'| > N$, the latter of which implies $n_\ell \geq \gamma|T_{\ell-1}'|/2\Delta - 1 > \gamma N/3\Delta$. It is left then only to show that $\mathbf{n}$ is $(\frac{\gamma}{4\Delta}, 2\gamma)$-descending.

For each $i \in [\ell - 1]$, we have $|T_{i+1}'| \leq \gamma|T_i'|$, so that $|T_i'| \geq |T_i| = 1 + |T_i'| - |T_{i+1}'| \geq 1 + (1-\gamma)|T_i'|$. Thus, for every $i \in [\ell]$, we have $(1-\gamma)|T_i'| \leq n_i \leq |T_i'|$. Then, for each $i \in [\ell - 1]$, we have $n_{i+1} \leq |T_{i+1}'| \leq \gamma|T_i'| \leq \gamma n_i/(1-\gamma) \leq 2\gamma n_i$. Furthermore, for each $i \in [\ell - 1]$, we have $n_{i+1} \geq (1-\gamma)|T_{i+1}'| \geq (1-\gamma)\gamma|T_i'|/2\Delta \geq (1-\gamma)\gamma n_i/2\Delta \geq \gamma n_i/4\Delta$. Thus, $\mathbf{n}$ is $(\frac{\gamma}{4\Delta}, 2\gamma)$-descending, as required. □

We will use Lemma 2.11 directly, as well as through the following corollary.

Corollary 2.12. *Let $0 < \gamma < 1/4$ and let $n, k, \Delta \in \mathbb{N}$ satisfy $n > (8\Delta/\gamma)^{k+1}$. Then for any n-vertex tree T with maximum degree at most Δ and any $t \in V(T)$, there exists a $(\frac{\gamma}{8\Delta}, 2\gamma)$-descending tuple $\mathbf{n} = (n_1, \ldots, n_k) \in \mathbb{N}^k$ such that T has an $\mathbf{n}$-decomposition $(T_1, \ldots, T_k)$ with $t \in V(T_1)$.*

Proof. Let $N = \frac{1}{2}(\frac{\gamma}{8\Delta})^k n$ and note that $N > \Delta/\gamma$. Then, by Lemma 2.11, there is a $(\frac{\gamma}{8\Delta}, \gamma)$-descending tuple $\mathbf{n}' = (n_1', \ldots, n_\ell') \in \mathbb{N}^\ell$ with $\frac{\gamma}{4\Delta}N \leq n_\ell \leq N$ such that T has an $\mathbf{n}$-decomposition $(T_1', \ldots, T_\ell')$. Now, as $N \geq |T_\ell'| - 1 \geq (\gamma/8\Delta)^\ell |T_1'| \geq (\gamma/8\Delta)^\ell n/2$, we have that $\ell \geq k$. Let $T_i = T_i'$ for each $i \in [k-1]$ and $T_k = \bigcup_{i=k}^{\ell} T_i'$. Let $n_i = n_i'$ for each $i \in [k-1]$, and $n_k = \sum_{i=k}^{\ell} n_i' - (\ell - k)$, and let $\mathbf{n} = (n_1, \ldots, n_k)$. Note that $\mathbf{n}$ is $(\frac{\gamma}{8\Delta}, 2\gamma)$-descending, and $(T_1, \ldots, T_k)$ is an $\mathbf{n}$-decomposition of T. □

2.6 Another expansion property

We will use the following variant of Proposition 2.6, which is proved in a similar manner.

Proposition 2.13. *Let $n_0, m \in \mathbb{N}$ and $d > 0$, and let G be an (m, n_0)-joined graph which contains a set $V \subset V(G)$ with $|V| \geq n_0 + (2d+2)m$. Then, there exists some $W \subset V(G)$ such that $|W| < m$ and, for each $U \subset V(G) \setminus W$ with $|U| \leq m$, $|N_G(U, V \setminus W)| \geq d|U|$.*

Proof. Let $W \subset V$ be a maximal subset subject to $|W| < 2m$ and $|N_G(W, V)| \leq d|W|$, possible as $W = \emptyset$ satisfies these conditions. Note that if $|W| \geq m$, then, as G is (m, n_0)-joined, $|V \setminus (W \cup N_G(W)| < n_0$, so that $|N_G(W, V)| \geq |V| - |W| - n_0 \geq 2dm \geq d|W|$, a contradiction. Thus, we have $|W| < m$.

Now, if $U \subset V(G) \setminus W$ satisfies $|U| \leq m$ and $U \neq \emptyset$, then, as $|W| < m$, we have $|U \cup V| < 2m$ and $|U \cup W| > |W|$. To avoid contradicting the choice of W, we must have $|N_G(U \cup W, V)| \geq d|U \cup W|$, and hence $|N_G(U, V \setminus W)| \geq d|U \cup W| - d|W| \geq d|U|$. Thus, for each $U \subset V(G)$ with $|U| \leq m$, $|N_G(U, V \setminus W)| \geq d|U|$. □

2.7 Tree embeddings and (d, m)-extendability

As noted in Section 2.2, to carry out our tree embedding schemes we use techniques that come from combining a development by Haxell [17] of a tree embedding method of Friedman and Pippenger [13] with a 'roll-back' idea of Johannsen (see [10]) as used by Draganic, Krivelevich and Nenadov [10]. We use the language of extendability (first used by Glebov, Johannsen, and Krivelevich [14]): in a graph with certain expansion conditions we embed a tree iteratively, each time adding a leaf so that the embedded subgraph maintains an 'extendability condition'. Key to the flexibility of this method is that removing a leaf ('rolling-back') maintains the extendability condition.

The key definition here is that of a *(d, m)-extendable subgraph S* in a graph G, which is a subgraph with the property that every small subset of $V(G)$ has many neighbours outside S. Note that the following definition uses the set $N'_G(U) = \cup_{v \in U} N_G(v)$.

Definition 2.14. Let $d, m \in \mathbb{N}$ be such that $d \geq 3$ and $m \geq 1$. Let G be a graph and let $S \subset G$ be a subgraph. We say that S is *(d, m)-extendable in G* if S has maximum degree at most d and, for all $U \subset V(G)$ with $1 \leq |U| \leq 2m$, we have

$$|N'_G(U) \setminus V(S)| \geq (d-1)|U| - \sum_{u \in U \cap V(S)} (d_S(u) - 1). \tag{2.1}$$

The next lemma shows how to verify the extendability condition using only the external neighbourhood.

Proposition 2.15. *Let $d, m \in \mathbb{N}$ be such that $d \geq 3$ and $m \geq 1$. Let G be a graph and let $S \subset G$ be a subgraph with maximum degree at most d. If, for all $U \subset V(G)$ with $1 \leq |U| \leq 2m$ we have $|N(U, V(G) \setminus V(S))| \geq d|U|$, then S is (d, m)-extendable.*

Proof. For each $U \subset V(G)$ with $1 \leq |U| \leq 2m$, we have

$$|N'(U) \setminus V(S)| \geq |N(U) \setminus V(S)| = |N(U, V(G) \setminus V(S))| \geq d|U| \geq (d-1)|U| - \sum_{u \in U \cap V(S)} (d_S(u) - 1),$$

as required. □

The next three lemmas are the core of the extendability method, and can be found as Lemmas 5.2.6, 5.2.7 and 5.2.8 in [14]. They will allow us to manipulate an extendable graph by adding/removing a vertex or an edge while remaining extendable. The latter two are simple to verify, while the first follows the original inductive step in the argument by Haxell [17]. The first lemma we state in our (m, m')-joined language, and it follows from Lemma 5.2.6 in [14] by observing that in such a graph G any set U of m vertices satisfies $|N_G(U)| \geq |G| - m' - |U| = |G| - m' - m$.

Lemma 2.16 (Adding a leaf)**.** *Let $d, m, m' \in \mathbb{N}$ be such that $d \geq 3$ and $m, m' \geq 1$, and let G be an (m, m')-joined graph. Let S be a (d, m)-extendable subgraph of G such that $|G| \geq |S| + (2d + 2)m + m' + 1$. Then, for every $s \in V(S)$ with $d_S(s) \leq d - 1$, there exists $y \in N_G(s) \setminus V(S)$ such that $S + sy$ is (d, m)-extendable.*

Lemma 2.17 (Removing a leaf)**.** *Let $d, m \in \mathbb{N}$ be such that $d \geq 3$ and $m \geq 1$, let G be a graph and let $S \subset G$ be a subgraph of G. Suppose that there exist vertices $s \in V(S)$ and $y \in N_G(s) \setminus V(S)$ so that $S + sy$ is (d, m)-extendable. Then, S is (d, m)-extendable.*

Lemma 2.18 (Adding an edge)**.** *Let $d, m \in \mathbb{N}$ be such that $d \geq 3$ and $m \geq 1$. Let G be a graph and let S be a (d, m)-extendable subgraph of G. If $s, t \in V(S)$ with $d_S(s), d_S(t) \leq d - 1$ and $st \in E(G)$, then $S + st$ is (d, m)-extendable in G.*

We will use Lemma 2.16 through the follow corollary where a tree is built on to an extendable subgraph by applying it iteratively (see also Corollary 3.7 in [21]).

Corollary 2.19. *Let $d, m, m' \in \mathbb{N}$ be such that $d \geq 3$, and let G be an (m, m')-joined graph. Let T be a tree with $\Delta(T) \leq d/2$ and let R be a (d, m)-extendable subgraph of G with maximum degree at most $d/2$. If $|R| + |T| \leq |G| - (2d + 2)m - m'$, then for every vertex $t \in V(T)$ and $v \in V(R)$, there is a copy S of T in $G - (V(R) \setminus \{v\})$ in which t is copied to v and $S \cup R$ is a (d, m)-extendable subgraph of G.*

2.8 Concentration results

We will use the following standard version of Chernoff's bound for binomial or hypergeometric random variables (see, e.g., [18] for the standard definition of such variables with parameters n and p and with parameters N, n and m, respectively).

Lemma 2.20 (see, e.g., Corollary 2.3 and Theorem 2.10 in [18])**.** *Let X be a hypergeometric random variable with parameters N, n and m, or a binomial random variable with parameters n and p. Then, for any $0 < \varepsilon \leq 3/2$,*

$$\mathbb{P}(|X - \mathbb{E}X| \geq \varepsilon \mathbb{E}X) \leq 2 \exp\left(-\frac{\varepsilon^2}{3}\mathbb{E}X\right).$$

A sequence of random variables $(X_i)_{i \geq 0}$ is a submartingale if $\mathbb{E}[X_{i+1} \mid X_0, \dots, X_i] \geq X_i$ for each $i \geq 0$. We will use the following Azuma-type bound for submartingales.

Lemma 2.21 (see, e.g., [26])**.** *Let $(X_i)_{i \geq 0}$ be a submartingale and let $c_i > 0$ for each $i \geq 1$. If $|X_i - X_{i-1}| < c_i$ for each $i \geq 1$, then, for each $n \geq 1$,*

$$\mathbb{P}(X_n - X_0 \leq t) \leq 2 \exp\left(-\frac{t^2}{2\sum_{i=1}^n c_i^2}\right).$$

3 Proof of Theorem 2.4

We now prove Theorem 2.4, which covers Theorem 2.2 when m is at most some large constant depending on Δ. As our n-vertex graph G here is $(m, \mu n)$-joined (for some small μ), sets with constant size (i.e., m) already have a neighbourhood covering almost all of the graph, making the graph very dense, with $\Theta(n^2/m)$ edges. Embedding a tree is not so difficult in this case, and, for example, we could use an approach of Balla, Pokrovskiy, and Sudakov [2] if T has linearly many leaves and an absorption approach like that in [21] if T does not have linearly many leaves. To take a unified approach, however, we will use an embedding inspired by work by the first author and Kathapurkar [19] and the first author and Böttcher, Parczyk, and Person [4].

To do this we first embed a small linear portion of the tree randomly, before extending this greedily vertex by vertex. Where a new leaf cannot be embedded because some embedded vertex, w say, has no more unused neighbours, we show that a vertex in the set U of (at least m) unused vertices can be swapped into the embedding to free up a neighbour of w, allowing the embedding to continue. This swapping property will follow from the random embedding of the first part of the tree, as the $(m, \mu n)$-joined property of G implies that for some $u \in U$, many *good* neighbours of w have a large common neighbourhood with u, so that, for many of these good neighbours, w' say, w' will have a vertex of T embedded to it with all of its neighbours in the embedding in T embedded to common neighbours of u and w' (see Claim 3.1) – exactly what we need to swap w' with u, and then use ww' to extend the embedding.

Proof of Theorem 2.4. To recap our situation, we have $\mu \ll 1/\Delta, 1/m$ with $m \leq \mu n$, an n-vertex $(m, \mu n)$-joined graph G, and an $(n-m+1)$-vertex tree T with $\Delta(T) \leq \Delta$ to embed into G. Note that, moreover, we can assume that $1/n \ll \mu$, as we could take some μ' with $\mu \ll \mu' \ll 1/\Delta, 1/m$ and prove this with μ' in place of μ first, where $m \leq \mu n$ implies that $1/n \leq \mu$ and thus $1/n \ll \mu'$.

Let γ and β satisfy $\mu \ll \gamma \ll \beta \ll 1/\Delta, 1/m$. Applying Proposition 2.10 with an arbitrary $t \in V(T)$, we find subtrees T_0 and T_1 of T sharing exactly one vertex t_1, such that $E(T) = E(T_1) \cup E(T_2)$ and $4\beta n \leq |T_0| \leq 8\Delta\beta n$. Label the remaining vertices in T as $t_2, \dots, t_{n-m+1}$ such that $T[\{t_1, \dots, t_i\}]$ is a tree for each $i \in [n-m+1]$, and $T[t_1, \dots, t_{|T_0|}] = T_0$. Moreover, do this so that, for a set $J \subset \{4, \dots, |T_0|\}$ with $|J| = \gamma n$, for each $j \in J$, $t_{j-3}t_{j-2}t_{j-1}t_j$ is a path in T and the neighbours of t_j in T except for t_{j-1} (if there are any) appear in the labelling directly after t_j, and the vertices t_j, $j \in J$ are all at distance at least 3 apart in T (that this is possible follows easily as $\Delta(T) \leq \Delta$ and $\gamma \ll \beta$).

Now, if $V(G)$ contains a set U of m vertices with degree at most $n/2m$, then $|N_G(U)| \leq m \cdot n/2m = n/2 < n - \mu n - m$, and hence G is not $(m, \mu n)$-joined, a contradiction. Therefore, setting V_0 to be the set of vertices in $V(G)$ with degree at least $n/2m$, we have $|V(G) \setminus V_0| < m$.

We now randomly embed T_0 into $G[V_0]$ vertex by vertex, as follows, creating an embedding ϕ. Arbitrarily, choose $v_1 \in V_0$ and set $\phi(t_1) = v_1$. Then, for each $2 \leq i \leq |T_0|$ in turn, let w_i be the image under ϕ of the sole neighbour of t_i in $T[\{t_1, \dots, t_i\}]$, choose $v_i \in N_G(w_i, V_0) \setminus \{v_1, \dots, v_{i-1}\}$ uniformly at random and set $\phi(t_i) = v_i$. Note that, for each $2 \leq i \leq |T_0|$, we have that $w_i \in V_0$, and therefore

$$|N_G(w_i, V_0) \setminus \{v_1, \dots, v_{i-1}\}| \geq \frac{n}{2m} - m - |T_0| \geq \frac{n}{2m} - m - 8\Delta\beta n \geq \frac{n}{4m},$$

so that this embedding successfully embeds T_0. We will show the following claim.

Claim 3.1. *For each $U \subset V(G)$ with $|U| = m$ and $v \in V_0 \setminus U$, with probability $1 - o(n^{-m-1})$, there are at least $4\mu n$ values of $j \in J$ such that $v_j v \in E(G)$ and $N_{\phi(T_0)}(v_j) \subset N_G(u)$ for some $u \in U$.*

Proof. Fix $U \subset V(G)$ with $|U| = m$ and $v \in V_0 \setminus U$. Now, for each $y \in V_0 \setminus U$, as $|N_G(y)| \geq n/2m$ and G is $(m, \mu n)$-joined, we have

$$|N_G(U) \cap N_G(y) \cap V_0)| \geq \frac{n}{2m} - m - \mu n - m \geq \frac{n}{4m}.$$

Thus, there is some $u \in U$ with $|N_G(u) \cap N_G(y) \cap V_0| \geq n/4m^2$. As this holds for every $y \in V_0 \setminus U$ and $|N_G(v, V_0) \setminus U| \geq n/2m - 2m \geq n/4m$, there is some $u \in U$ such that, for at least $n/4m^2$ vertices $y \in N(v, V_0) \setminus U$, we have $|N_G(u) \cap N_G(y) \cap V_0| \geq n/4m^2$. Say the set of these vertices y is $Y_{v,u}$.

We now show that, with probability at least $1 - o(n^{-(m+1)})$, for at least $4\mu n$ values of $j \in J$, we have $v_j \in Y_{v,u}$ and $N_{\phi(T_0)}(v_j) \subset N_G(u)$. Let $J = \{j_1, \dots, j_{\gamma n}\}$, in order, and, for each $i \in [\gamma n]$, let X_i be the indicator function for $v_{j_i} \in Y_{v,u}$ and $N_{\phi(T_0)}(v_{j_i}) \subset N_G(u)$.

Consider $i \in [\gamma n]$ and suppose we have embedded $t_1, \dots, t_{j_i - 3}$. As $v_{j_i-3} \in V_0$, for each $y \in Y_{v,u}$, note that, as $|N_G(u) \cap N_G(y) \cap V_0| \geq n/4m^2$, all but at most m vertices in $N_G(v_{j_i-3})$ have at least $n/8m^3$ neighbours in $|N_G(u) \cap N_G(y) \cap V_0|$, for otherwise, as before, we get that G is not $(m, n/8m^2)$-joined, a contradiction as $n/8m^2 \geq \mu n$. Therefore, the probability that $v_{j_i-1} \in N_G(u) \cap N_G(y) \cap V_0$, $v_{j_i} = y$ and any subsequent neighbours of t_{j_i} are embedded into $N_G(u) \cap N_G(y) \cap V_0$ is at least

$$\frac{|N_G(v_{j_i-3}, V_0) \setminus \{v_1, \dots, v_{j_i-3}\}| - m}{|N_G(v_{j_i-3}, V_0) \setminus \{v_1, \dots, v_{j_i-3}\}|} \cdot \frac{n/8m^3}{n} \cdot \frac{1}{n} \cdot \left(\frac{n/4m^2 - |T_0|}{n} \right)^{\Delta} \geq \frac{1}{n(8m)^{2\Delta+3}}.$$

As these events are distinct for each $y \in Y_{v,u}$, we have that, conditioned on any possible values of $v_1, \dots, v_{j_i-3}$,

$$\mathbb{P}(X_i = 1) \geq |Y_{v,u}| \cdot \frac{1}{n(8m)^{2\Delta+3}} \geq \frac{n}{4m^2} \cdot \frac{1}{n(8m)^{2\Delta+3}} \geq \frac{1}{(8m)^{2\Delta+5}}.$$

Therefore, setting $\alpha = \frac{1}{(8m)^{2\Delta+5}}$ so that $\alpha \gg \mu$, and letting $Z_i = \sum_{i'=1}^{i}(X_{i'} - \alpha)$ for each $i \in [\gamma n]$, we have that $(Z_0, Z_1, \dots, Z_{\gamma n})$ is a submartingale, using that, for all $1 \leq i < i' \leq \gamma n$, t_{j_i} and $t_{j_{i'}}$ have distance at least 3 apart in T, and t_{j_i} and its neighbours appear in the sequence $t_1, \dots, t_{|T_0|}$ before $t_{j_{i'}}$ or any of its neighbours. Furthermore, for each $i \in [\gamma n]$, we have $|Z_i - Z_{i-1}| \leq 1$. Therefore, by Azuma's inequality (Lemma 2.21), we have

$$\mathbb{P}\Big(\sum_{i \in [\gamma n]} (X_i - \alpha) \leq \alpha\gamma n/2\Big) \leq 2\exp\Big(-\frac{(\alpha\gamma n/2)^2}{2\gamma n}\Big) = 2\exp(-\alpha^2\gamma n/8),$$

so that $\sum_{i\in[\gamma n]} X_i \geq \alpha\gamma n/2 \geq 4\mu n$ with probability at least $1 - 2\exp(-\alpha^2\gamma n/8) = 1 - o(n^{-(m+1)})$, as required, using $\alpha, \gamma \gg \mu \gg 1/n$. $\square$

Therefore, by Claim 3.1 and a union bound, we can assume that ϕ has the property that, for each $U \subset V(G)$ with $|U| = m$ and $v \in V_0 \setminus U$, there are at least $4\mu n$ values of $i \in J$ such that $v_i v \in E(G)$ and $N_{\phi(T_0)}(v_i) \subset N_G(u)$ for some $u \in U$. Next, we greedily extend the embedding of T_0 to one of T, where if we cannot simply extend the embedding by embedding t_i we use the property from Claim 3.1 to swap a single vertex in the embedding to allow this extension. That is, we do the following.

Let $\phi_{|T_0|} = \phi$. Then, for each i with $|T_0| < i \leq n - m + 1$ in turn, we will take ϕ_{i-1}, an embedding of $T[\{t_1, \dots, t_{i-1}\}]$ into $G[\{v_1, \dots, v_{i-1}\}]$, and find $v_i \in V(G) \setminus \{v_1, \dots, v_{i-1}\}$ and an embedding ϕ_i of $T[\{t_1, \dots, t_i\}]$ into $G[\{v_1, \dots, v_i\}]$, as follows. Let w_i be the image under ϕ_{i-1} of the sole neighbour of t_i in $T[\{t_1, \dots, t_i\}]$ and,

i) if possible, choose $v_i \in N_G(w_i, V_0 \setminus \{v_1, \dots, v_{i-1}\})$ and set $\phi_i(t_i) = v_i$ and $\phi_i(t_j) = \phi_{i-1}(t_j)$ for each $j < i$,

ii) otherwise, if possible, pick $v_i \in V(G) \setminus \{v_1, \dots, v_{i-1}\}$ and $j \in J$ such that $\phi_{i-1}(t_j) = v_j$, $w_i v_j \in E(G)$ and $\phi_{i-1}(N_{T_0}(t_j)) \subset N_G(v_i)$, and set $\phi_i(t_i) = \phi_{i-1}(t_j)$, $\phi_i(t_j) = v_i$ and $\phi_{i-1}(t_{j'}) = \phi_{i-1}(t_{j'})$ for each $j' \in [i-1] \setminus \{j\}$ (noting that this does embed $T[\{t_1, \dots, t_i\}]$ into $G[\{v_1, \dots, v_i\}]$ as $N_{T_0}(t_j) = N_T(t_j)$), and,

iii) otherwise, stop the embedding.

Suppose the process first fails to embed t_ℓ for some $|T_0| < \ell \leq |T|$, otherwise we have embedded all of T in G and are done. We first show that step ii) has been carried out at most $3\mu n$ times. Indeed, suppose otherwise, and let I be the set of the *first* μn values of i with $|T_0| < i < \ell$ for which v_i was embedded using step ii), so that $i < \ell - 2\mu n$ for each $i \in I$. Now, for each i with $|T_0| < i < \ell$, ϕ_{i-1} and ϕ_i differ on at most one vertex in $\{t_1, \dots, t_{i-1}\}$ which (if it exists) must be some vertex t_j with $j \in J$ for which we still have $\phi_{i-1}(t_j) = v_j$. Therefore, as $w_i \in \{v_1\} \cup \phi_{i-1}(V(T) \setminus V(T_0))$, once a vertex has been embedded to w_i by some $\phi_{i'}$, it is embedded to w_i by any subsequent embedding. Thus, as $\Delta(T) \leq \Delta$, there must be at most Δ values of $i' > |T_0|$ with $w_{i'} = w_i$. Therefore, letting $W = \{w_i : i \in I\}$, we have $|W| \geq |I|/\Delta$. Let $V_1 = V_0 \setminus \{v_1, \dots, v_{\ell-2\mu n}\}$, and note that there are no edges in G between W and V_1, for otherwise there is an edge from w_i to V_1 for some $i \in I$, which contradicts that t_i was not embedded by a step i). Thus, as $|V_1| \geq 2\mu n - m \geq \mu n$ and $|W| \geq \mu n/\Delta \geq m$ (using $m \ll 1/\mu \ll n$), this contradicts that G is $(m, \mu n)$-joined. Thus, we have that step ii) has been carried out at most $3\mu n$ times.

Let then $J_0 = \{j \in J : \phi_{\ell-1}(v_j) = \phi(v_j)\}$, so that $|J \setminus J_0| \leq 3\mu n$. Now, we have $w_\ell \in \{v_1\} \cup \phi_{\ell-1}(V(T) \setminus V(T_0))$, and from the process above, we have that $\phi_{\ell-1}(t_1) = v_1$ and, for each $|T_0| < i < \ell$, t_i is either embedded into V_0 at step i) or embedded to some vertex $\phi_{i-1}(t_j) = \phi(t_j)$, which is also in V_0. Thus, $w_\ell \in V_0$. Let $U_\ell \subset V(G) \setminus \{v_1, \dots, v_{\ell-1}\}$ have size m, possible as $|V(G) \setminus \{v_1, \dots, v_{\ell-1}\}| \geq n - (|T| - 1) \geq m$.

Then, noting $w_\ell \in V_0 \setminus U_\ell$ and using the property in Claim 3.1, there is some $j \in J_0$ and $v_\ell \in U_\ell$ with $w_\ell v_j \in E(G)$ and $\phi(N_{T_0}(t_j)) \subset N_G(v_\ell)$. As the vertices in J have distance at least 3 apart in the tree, for each j' with $t_{j'} \in N_{T_0}(t_j)$, we have $j' \notin J$, and therefore $\phi_{\ell-1}(N_{T_0}(t_j)) = \phi(N_{T_0}(t_j)) \subset N_G(v_\ell)$. Note that v_ℓ and j show that ii) can be carried out to embed t_ℓ, contradiction. Therefore, the process runs until all the vertices of T have been embedding, showing that G contains a copy of T. □

4 Embedding trees to cover vertex subsets

In this section, we prove the following key embedding result, which we use in Section 5 to embed each piece of the tree into the vortex while covering a prescribed set of vertices, represented in Lemma 4.1 by the set X.

Lemma 4.1. *Let $\Delta \geq 2$, $d \geq 20$, $m \in \mathbb{N}$ and $0 < \gamma < 1/10$ satisfy $m \geq d^8$, $d \geq \Delta$ and $d \gg \Delta^3/\gamma$. Let G be an m-joined graph containing a vertex v. Let T be a tree satisfying $|T| \geq 2d^2m$ and $\Delta(T) \leq \Delta$, and let $t \in V(T)$. Suppose $X \subset V(G) \setminus \{v\}$ contains at most $(1-\gamma)|T|$ vertices, and $I(X \cup \{v\})$ is (d,m)-extendable in G. Let t' be a leaf of T which is not t, and suppose $|G| \geq |T| + 20dm$.*

Then, G contains a copy of T that covers X, in which t is copied to v, and t' is copied into $V(G) \setminus X$.

In order to prove this, we first show the following result, which embeds a tree to cover most of a prescribed set of vertices.

Lemma 4.2. *Let $d, m, \Delta \in \mathbb{N}$ satisfy $d \geq 3$ and $d \geq \Delta$. Let G be an m-joined graph which contains a set $X \subset V(G)$ and a vertex $v \in V(G) \setminus X$ such that $I(X \cup \{v\})$ is (d,m)-extendable in G. Let T be a tree with $\Delta(T) \leq \Delta$ and let $t \in V(T)$. Suppose that*

$$|T| \geq |X| + \Delta m + 2 \quad \text{and} \quad |G| \geq |T| + (2d+4)m + 1. \tag{4.1}$$

Then, there is a copy S of T such that t is copied to v, $S \cup I(X)$ is (d,m)-extendable in G, and $|X \setminus V(S)| < m$.

Proof. Let $\ell = |T|$. Let $t_1 = t$, and label the vertices of $V(T) \setminus \{t\}$ as $t_2, \ldots, t_\ell$ so that, for each $i \in [\ell]$, $T_i = T[\{t_1, \ldots, t_i\}]$ is a tree. For each $2 \leq i \leq \ell$, let s_i be the unique neighbour of t_i in $T[\{t_1, \ldots, t_i\}]$. Let $v_1 = v$ and let S_1 be the graph containing just the vertex v_1, so that S_1 is a copy of T_1 with t_1 copied to v_1, and $S_1 \cup I(X)$ is (d,m)-extendable in G. Now, carry out the following process, where for each $j = 2, \ldots, \ell$ in turn, if possible we perform the following Step $\mathbf{C}_j$ to embed t_j and produce a copy S_j of T_j in which t is copied to v and $S_j \cup I(X)$ is (d,m)-extendable in G.

$\mathbf{C}_j$
- Let u_j be the copy of s_j in S_{j-1}. If there exists a vertex $v_j \in X \setminus V(S_{j-1})$ so that $u_j v_j \in E(G)$, then let $S_j = S_{j-1} + u_j v_j$ and note that S_j is a copy of T_j in G. Moreover, $S_j \cup I(X)$ is (d,m)-extendable in G by Lemma 2.18. In this case, we say Step $\mathbf{C}_j$ is a *good* step.
- Otherwise, if possible let v_j be a neighbour of u_j in $V(G) \setminus (V(S_{j-1}) \cup X)$ such that, setting $S_j = S_{j-1} + u_j v_j$, $S_j \cup I(X)$ is (d,m)-extendable in G. Note that S_j is a copy of T_j in G. In this case, we say Step $\mathbf{C}_j$ is a *neutral* step.

We will show that we can successfully perform steps $\mathbf{C}_j$ for each $2 \leq j \leq \ell$, and that the resulting tree S_ℓ will satisfy $|X \setminus V(S_\ell)| < m$. Then, as S_ℓ is a copy of $T_\ell = T$ with $t_1 = t$ copied to $v_1 = v$, this completes the proof.

Claim 4.3. *For each $2 \leq j \leq \ell$, if we have reached the start of step $\mathbf{C}_j$ and $|X \setminus V(S_{j-1})| \geq m$, then fewer than Δm neutral steps have been taken so far.*

Proof of Claim 4.3. Suppose we have reached the start of step $\mathbf{C}_j$ and $|X \setminus V(S_{j-1})| \geq m$. Let $J \subset [j-1]$ be the set of indices i for which the step $\mathbf{C}_i$ was a neutral step, noting that, for each $i \in J$, u_i has no neighbour in G in $X \setminus V(S_{i-1})$, and hence no neighbour in $X \setminus V(S_{j-1})$. Thus, as G is m-joined, we have $|\{u_i : i \in J\}| < m$. As $\Delta(T) \leq \Delta$, $|\{u_i : i \in J\}| \geq |J|/\Delta$, so $|J| < \Delta m$, and thus before the start of step $\mathbf{C}_j$ we took fewer than Δm neutral steps. ⊡

Suppose then, for contradiction, that the process reaches the start of Step $\mathbf{C}_j$ for some $2 \leq j \leq \ell$ and fails to take either a good or neutral step. Note that the number of good steps that have been taken is $|V(S_{j-1}) \cap X|$ and the number of neutral steps that have been taken is $|V(S_{j-1}) \setminus X| - 1$, which is at most $|S_{j-1}| - |X| + m - 2$ if $|X \setminus V(S_{j-1})| < m$. Therefore, by Claim 4.3, the number of neutral steps is at most $\max\{\Delta m, |S_{j-1}| - |X| + m - 2\}$, so that

$$\begin{aligned}|S_{j-1} \cup I(X)| = (|X| + 1) + (|V(S_{j-1}) \setminus X| - 1) &\leq |X| + 1 + \max\{\Delta m, |S_{j-1}| - |X| + m - 2\} \\ &\leq \max\{|X| + \Delta m + 1, |S_{j-1}| + m\} \overset{(4.1)}{\leq} |T| + m.\end{aligned}$$

Therefore, by (4.1), $|G| \geq |T| + (2d+4)m + 1 \geq |S_{j-1} \cup I(X)| + (2d+2)m + m + 1$, so that, by Lemma 2.16, there is some $v_j \in V(G) \setminus (V(S_{j-1}) \cup X)$ which is a neighbour of u_j in G and such that $S_{j-1} \cup I(X) + u_j v_j$ is (d, m)-extendable in G, contradicting that we did not take a good or neutral step at Step $\mathbf{C}_j$.

Therefore, the process has successfully taken Step $\mathbf{C}_j$ for each $2 \leq j \leq \ell$, and produced S_ℓ, a copy of T_ℓ in G with $t_1 = t$ copied to $v_1 = v$. Finally, note that at the start of Step $\mathbf{C}_\ell$ we will have taken at most $|X|$ good steps, and thus at least $(\ell - 2) - |X| = |T| - |X| - 2 \geq \Delta m$ neutral steps by (4.1). Therefore, by Claim 4.3, $|X \setminus V(S_\ell)| \leq |X \setminus V(S_{\ell-1})| < m$, and thus S_ℓ is the required copy of T. □

Given a graph G and a subset $Q \subset V(G)$, we say that Q is k-separated in G if each pair of vertices in Q is at distance at least k in G. We now need two results from [21], which we state in a slightly simpler form that follow directly from [21, Corollary 3.16] and [21, Lemma 4.1], respectively.

Proposition 4.4. *Let $k \geq 0$ and $\Delta \geq 2$. Let T be a tree with $\Delta(T) \leq \Delta$ and $|T| \geq 3\Delta^k$. Then, there is a subset $Q \subset V(T)$ which is $(2k+2)$-separated in T such that $|Q| \geq |T|/(8k+8)\Delta^k$.*

Lemma 4.5. *Let $k, d, m \in \mathbb{N}$ with $d \geq 20$. Let G be an m-joined graph and let $R \subset G$ be a subgraph with $\Delta(R) \leq d/4$. Suppose $X \subset V(G) \setminus V(R)$ is such that $R \cup I(X)$ is (d, m)-extendable in G. Let T be a tree with $\Delta(T) \leq d/4$ which has a set of $3|X|$ vertices which is $(4k+4)$-separated in T. Let $t \in V(T)$ and $r \in V(R)$, and suppose $|R| + |X| + |T| \leq |G| - 10dm - 2k$.*

Then, there is a copy S of T in $G - V(R \setminus \{r\})$ so that t is copied to r, $R \cup I(X) \cup S$ is (d, m)-extendable in G and $|X \setminus V(S)| \leq 2m/(d-1)^k$.

We can now prove Lemma 4.1. The proof takes a tree decomposition using Corollary 2.12, embed the first piece to cover most of X using Lemma 4.5, and then repeatedly uses Proposition 4.4 and Lemma 4.5 to embed the remaining pieces while covering more and more of the uncovered vertices in X until all the vertices in X are covered.

Proof of Lemma 4.1. Let $\gamma_2 = \gamma/10$, $\gamma_1 = \gamma_2/16\Delta$ and $T' = T - t'$. Let ℓ be the smallest integer such that $2m/(d-1)^{\ell-1} < 1$, so that $\ell \geq 9$ as $m \geq d^8$. By minimality, we also have that $2m \geq (d-1)^{\ell-2}$, and thus, using $d \gg \Delta^3/\gamma$, we have

$$|T'| \geq 2d^2 m - 1 \geq d^2(d-1)^{\ell-2} - 1 > (16\Delta/\gamma_2)^{\ell+1}. \tag{4.2}$$

Then, by Corollary 2.12, there is a (γ_1, γ_2)-descending tuple $\mathbf{n} = (n_1, \cdots, n_\ell) \in \mathbb{N}^\ell$ such that T' has an $\mathbf{n}$-decomposition $(T_1, \ldots, T_\ell)$ with $t \in V(T_1)$. Let $t_0 = t$ and, for each $i \in [\ell - 1]$, let t_i be the unique vertex shared by T_i and T_{i+1}.

Note that $\sum_{i=2}^{\ell} n_i \leq 2\gamma_2 n_1 \leq 2\gamma_2 |T|$, so

$$|T_1| \geq (1 - 2\gamma_2)|T| - 1 \geq (1 - \gamma/2)|T| \geq |X| + \gamma|T|/2 \geq |X| + \Delta m + 3.$$

As $|G| \geq |T| + 20dm$, we may apply Lemma 4.2 to find a copy S_1 of T_1 in G in which t is copied to v, $S_1 \cup I(X)$ is (d, m)-extendable in G and $|X \setminus V(S_1)| < m$. Next, for each $2 \leq j \leq \ell$ in turn, do the following.

- $\mathbf{D}_j$ If possible, find a copy S_j of T_j in $G - (\cup_{i \in [j-1]} V(S_i) \setminus \{v_{j-1}\})$ such that t_{j-1} is copied to v_{j-1}, $(\cup_{i \in [j]} S_i) \cup I(X)$ is (d, m)-extendable in G, and $|X \setminus (\cup_{i \in [j]} S_i)| < 2m/(d-1)^{j-1}$.

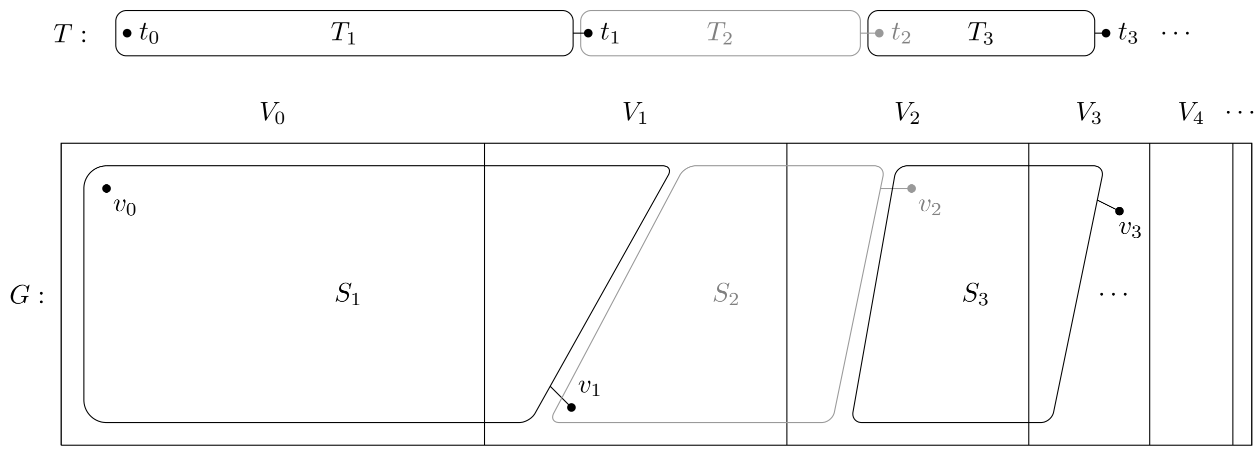

Figure 1: To embed T for Theorem 2.5, we divide T into $T_1 \cup \ldots \cup T_\ell$ using Lemma 2.11, then find an accompanying vortex partition $V_0 \cup V_1 \cup \ldots \cup V_\ell$ (of a large subgraph G' of G) using Lemma 5.8. We then use Lemma 4.1 to iteratively embed T_i into $G[V_{i-1} \cup V_i]$, where, if $i < \ell$, any unused vertices in V_{i-1} are covered and the common vertex t_i of T_i and T_{i+1} is embedded to v_i in V_i.

Suppose that for some $2 \leq j \leq \ell$ this was not possible. Note that the set $X_j = X \setminus (\cup_{i\in[j-1]} S_i)$ satisfies $|X_j| < 2m/(d-1)^{j-2}$, and $|T_j| = n_j + 1 \geq \gamma_1^{j-1} n_1 \geq \gamma_1^{j-1}|T|/2 \geq \gamma_1^{j-1} d^2 m$. Hence, we may apply Proposition 4.4 to obtain a set Q_j of vertices in T_j that is $(4j)$-separated, with

$$|Q_j| \geq \frac{|T_j|}{16j\Delta^{2j-1}} \geq \frac{\gamma_1^{j-1} d^2 m}{16j\Delta^{2j-1}} \geq \frac{6m}{(d-1)^{j-2}} \cdot \frac{\gamma_1^{j-1}(d-1)^j}{96j\Delta^{2j-1}} \geq \frac{6m}{(d-1)^{j-2}} \geq 3|X_j|,$$

where the second last inequality follows from $d \gg \Delta^3/\gamma$. Furthermore, we have

$$|\cup_{i\in[j-1]} S_i| + |X_j| + |T_j| \leq |T| + |X_j| \leq |T| + 2m \leq |G| - 10dm - 2(j-1),$$

so we may apply Lemma 4.5 with $R = \cup_{i\in[j-1]} S_i$, $X = X_j$, $T = T_j$, $r = v_{j-1}$ and $t = t_{j-1}$ to find a copy S_j of T_j, satisfying all the required conditions in $\mathbf{D}_j$, which is a contradiction.

Therefore, we can complete $\mathbf{D}_j$ for each $2 \leq j \leq \ell$. Taking $S' = \cup_{j\in[\ell]} S_j$, we have, then, a copy of T' in which t is copied to v such that $I(X) \cup S'$ is (d,m)-extendable and $|X \setminus V(S')| < 2m/(d-1)^{\ell-1} < 1$. In other words, the copy S' of T' covers X. Let s' be the vertex of S' which needs a leaf added to make S' into a copy of T. As $|G| \geq |S'| + (2d+2)m + m + 1$, by Lemma 2.16 there is some $v' \in V(G) \setminus V(S') \subset V(G) \setminus X$ which is a neighbour of s'. Let $S = S' + s'v'$, and note that this is a copy of T which covers X, in which t is copied to v and t' is copied into $V(G) \setminus X$, as required. □

5 Proof of Theorem 2.5

In this section, we prove Theorem 2.5, where we embed a tree T with $n - m + 1$ vertices and $\Delta(T) \leq \Delta$ into an n-vertex $(m, \mu n)$-joined (D, m)-expander graph G, where $m \leq \mu n$ and $\mu, 1/D, 1/m \ll 1/\Delta$. We wish to do the following. First, we partition T into trees $T_1 \cup \ldots \cup T_\ell$ with geometrically-decreasing size. We then use the sizes of the trees in this partition to inform the sizes of a partition of all but at most $m/4$ vertices of G as $V_0 \cup V_1 \cup \ldots \cup V_\ell$. Then, for each $i \in [\ell]$ in turn, we embed T_i into $G[V_{i-1} \cup V_i]$, attached appropriately to the existing embedding, so that, if $i < \ell$, all the unused vertices in V_{i-1} are covered, and the sole vertex shared by T_i and T_{i+1} is embedded into V_i. This is depicted in Figure 1.

To embed each T_i, except for the last piece, T_ℓ, where we have very few spare vertices, we use Lemma 4.1. In order to do this in the manner described for the partition $V_0 \cup V_1 \cup \ldots \cup V_\ell$, we need several properties, which we record, as follows, as a vortex-partition. Essentially, we require the sets to have (approximately) the right size (see **E1**), and that, for each $i \in [\ell]$, the subgraph we are using to embed T_i (which will be a subgraph of $G[V_{i-1} \cup V_i]$) is joined (see **E2**) and the set of any remaining vertices in V_{i-1} to be covered is, considered as a empty graph, extendable (which will follow from **E3**).

Definition 5.1 (Vortex partition)**.** Let $d \in \mathbb{N}$ and $\lambda > 0$, and let $\mathbf{n} \in \mathbb{N}^{\ell+1}$. For a graph G an $(\mathbf{n}, \lambda, d)$-vortex-partition in G is a partition $V_0 \cup V_1 \cup \ldots \cup V_\ell$ of $V(G)$ such that the following hold.

E1 For each $i \in [\ell]_0$, $|V_i| = (1 \pm \lambda)n_i$.

E2 For each $i \in [\ell]$, $G[V_{i-1} \cup V_i]$ is (λn_{i-1})-joined.

E3 For each $i \in [\ell]$, $I(V_{i-1})$ is $(d, \lambda n_{i-1})$-extendable in $G[V_{i-1} \cup V_i]$.

We need the vortex partition that we find to cover all but very few (at most $m/4$) vertices in the graph G. To do this, we first find an initial 'vortex' in Section 5.1, before 'cleaning' this into a vortex partition in Section 5.2. We can then carry out our embedding in Section 5.3, completing the proof of Theorem 2.5.

5.1 Vortices

To find our vortex partition, we first find a vortex in our graph G, which is a *nested* sequence of vertex sets with an increasingly good expansion property as the sets decrease in size. We define this precisely as follows.

Definition 5.2 (Vortex)**.** Let ℓ be a non-negative integer, $\lambda > 0$, $m \in \mathbb{N}$, and let $\mathbf{n} = (n_0, \ldots, n_\ell) \in \mathbb{N}^{\ell+1}$. For a graph G on n_0 vertices, an $(\mathbf{n}, m, \lambda)$-vortex in G is a sequence of subsets $U_0 \supset U_1 \supset \ldots \supset U_\ell$ such that $U_0 = V(G)$ and, for each $i \in [\ell]$,

F1 $|U_i| = n_i$, and

F2 every subset $U \subset U_{i-1}$ with $|U| = m$ satisfies $|N(U, U_i)| \geq (1 - \lambda)|U_i|$.

For an appropriate descending tuple $\mathbf{n} = (n_0, \ldots, n_\ell) \in \mathbb{N}^{\ell+1}$ and an n-vertex graph G which is $(m, \mu n)$-joined we will find an $(\mathbf{n}, m, 2\lambda)$-vortex $U_0 \supset U_1 \supset \ldots \supset U_\ell$ in G (with $\mu \ll \lambda$) by randomly choosing such a nested subsequence subject only to $|U_i| = n_i$ for all $i \in [\ell]_0$, and showing that it is an $(\mathbf{n}, m, 2\lambda)$-vortex with high probability (see Lemma 5.3). In this we are inspired by a similar vortex used by Barber, Kühn, Lo, and Osthus [3] in a graph with high minimum degree, and we use some similar calculations in the analysis.

More challengingly though, we want to show that we can ensure that $G[U_{\ell-1}]$ is also (λm)-joined. To this end, we additionally choose the vortex with two vertex sets $V_0 \supset V_1$ such that, with high probability, V_0 and V_1 fit into the randomly chosen nested subsequence that will be (with high probability) our vortex so that V_1 contains $U_{\ell-1}$. Then, we show that with high probability, every m-set in V_0 has at most λm non-neighbours in V_0, and, conditioned on this, that every (λm)-set in V_1 has at most λm non-neighbours in V_0. For this, we need the random sets to jump down in size by more than the standard ratio between the sets in the vortex, which is why we take the additional sets V_0 and V_1. We also use sets W and W' chosen disjointly which will fit within $U_1 \setminus U_2$ with high probability, for a similar ease of analysis for the expansion of small sets into $U_1 \setminus U_2$. Using all this, we prove the following lemma finding the vortex that we will need.

Lemma 5.3. *Let*

$$\frac{1}{m} \ll \mu \ll \frac{1}{K}, \frac{1}{D} \ll \lambda, \gamma_1, \gamma_2 \leq \frac{1}{9},$$

with $\gamma_1 < \gamma_2$ and let n be such that $m \leq \mu n$. Let G be an n-vertex $(m, \mu n)$-joined graph which is a (D, m)-expander, and let $\mathbf{n} = (n_0, n_1, \ldots, n_\ell)$ be a (γ_1, γ_2)-descending tuple with $n_0 = n$ and $\gamma_1 Km \leq n_\ell \leq 2Km$.

Then, G contains an $(\mathbf{n}, m, 2\lambda)$-vortex $U_0 \supset U_1 \supset \ldots \supset U_\ell$ such that $G[U_{\ell-1}]$ is λm-joined and, for every $U \subset V(G)$ with $|U| = \lfloor \frac{m}{4} \rfloor$, $|N_G(U, U_1 \setminus U_2)| \geq \gamma_1 Dm/200$.

Proof. Let K_0 be such that $\mu \ll \frac{1}{K_0} \ll \frac{1}{K}$ and let $k \in [\ell]$ be some index with $\gamma_1 K_0 m \leq n_k \leq K_0 m$. Let $p_0 = \frac{2n_k}{n_0}$ and $q = \frac{n_1 - n_2}{4n_0}$, and take random disjoint sets V_0, W and W' in $V(G)$ so that the location of each vertex is selected uniformly at random and is in V_0 with probability p_0, in W with probability q and in W' with probability q. Let $V_1 \subset V_0$ be formed by including each element of V_0 independently at random with probability $p_1 := \frac{2n_{\ell-1}}{n_k}$. Let F be the event that $n_k \leq |V_0| \leq 3n_k$, $n_{\ell-1} \leq |V_1| \leq 9n_{\ell-1}$, and $\frac{n_1-n_2}{6} \leq |W|, |W'| \leq \frac{n_1-n_2}{3}$. If F holds, then let $U_0 \supset U_1 \supset \ldots \supset U_\ell$ be a sequence of sets chosen uniformly at random subject to $U_{k-1} \supset V_0 \supset U_k$, $U_{\ell-2} \supset V_1 \supset U_{\ell-1}$, $U_1 \setminus U_2 \supset W \cup W'$, and $|U_i| = n_i$ for each $i \in [\ell]_0$ (where we have used that $|W \cup W'| \leq \frac{2(n_1-n_2)}{3}$, $|V_1| \leq 9n_{\ell-1} \leq n_{\ell-1}/\gamma_2 \leq n_{\ell-2}$, and, similarly, $|V_0| \leq n_{k-1}$). If F does not hold, then let $U_0 \supset U_1 \supset \ldots \supset U_\ell$ be a sequence of sets chosen uniformly at random subject to $U_i = n_i$ for each $i \in [\ell]_0$.

Claim 5.4. *With probability more than* $3/4$*,* F *holds.*

Claim 5.5. *With probability more than* $3/4$*,* $U_0 \supset U_1 \supset \ldots \supset U_\ell$ *is an* $(\mathbf{n}, m, 2\lambda)$*-vortex in* G*.*

Claim 5.6. *With probability more than* $3/4$*, for every set* $U \subset V(G)$ *with* $|U| = \lfloor \frac{m}{4} \rfloor$*, we have* $|N_G(U, W \cup W')| \geq \gamma_1 Dm/200$.

Claim 5.7. *With probability more than* $3/4$*,* $G[V_1]$ *is* (λm)*-joined.*

These claims easily imply the lemma. Indeed, by them, we have that, with positive probability, F holds, $U_0 \supset U_1 \supset \ldots \supset U_\ell$ is an $(\mathbf{n}, m, 2\lambda)$-vortex in G, every subset $U \subset V$ with $|U| = \lfloor \frac{m}{4} \rfloor$ satisfies $|N(U, W \cup W')| \geq \gamma_1 Dm/200$, and $G[V_1]$ is (λm)-joined. Moreover, as F holds, we have $U_{\ell-1} \subset V_1$ and thus $G[U_{\ell-1}]$ is (λm)-joined, and, as $W \cup W' \subset U_1 \setminus U_2$, we have that every subset $U \subset V(G)$ with $|U| = \lfloor \frac{m}{4} \rfloor$ satisfies $|N(U, U_1 \setminus U_2)| \geq \gamma_1 Dm/200$, as required. Thus, it is left only to prove the four claims.

Proof of Claim 5.4. Now, by Lemma 2.20, using $p_0 = 2n_k/n_0$ and $n_k \geq \gamma_1 K_0 m$, we have

$$\mathbb{P}\left(\big||V_0| - 2n_k\big| > n_k\right) \leq 2\exp(-2n_k/12) = 2\exp(-n_k/6) < 1/16.$$

Conditioning on $n_k \leq |V_0| \leq 3n_k$ and again by Lemma 2.20, using $p_1 = 2n_\ell/n_k$ and $n_\ell \geq \gamma_1 Km$, we have

$$\mathbb{P}(|V_1| \leq n_\ell \text{ or } |V_1| \geq 9n_\ell) \leq \mathbb{P}(||V_1| - p_1|V_0|| \geq p_1|V_0|/2) \leq 2\exp(-n_\ell/6) < 1/16.$$

Similarly, by Lemma 2.20, using $q = \frac{n_1-n_2}{4n_0}$ and $n_2 \leq \gamma_2 n_1$, we have for each $i \in [2]$,

$$\mathbb{P}\left(\big||W_i| - \tfrac{n_1-n_2}{4}\big| > \tfrac{n_1-n_2}{12}\right) \leq 2\exp(-(n_1-n_2)/108) < \frac{1}{16}. \tag{5.1}$$

Thus, with probability more than $1 - 4/16 = 3/4$, we have $n_k \leq |V_0| \leq 3n_k$, $n_\ell \leq |V_1| \leq 9n_\ell$, and $\frac{n_1-n_2}{6} \leq |W|, |W'| \leq \frac{n_1-n_2}{3}$, and hence F holds. ⊡

Proof of Claim 5.5. Note that, as V_0, V_1, W, W' are themselves random sets, the sets $U_0 \supset U_1 \supset \ldots \supset U_\ell$ have the same distribution as a nested sequence of sets chosen uniformly at random subject to $|U_i| = n_i$ for each $i \in [\ell]_0$, whether F holds or not. Now, for each $i \in [\ell]_0$, let $\varepsilon_i = (m/n_i)^{1/4}$ and $\mathbf{n}_i = (n_0, n_1, \ldots, n_i)$. For each $i \in [\ell]$, let E_i be the event that $U_0 \supset U_1 \supset \ldots \supset U_{i-1}$ is an $(\mathbf{n}_i, m, \lambda + \varepsilon_{i-1})$-vortex, but $U_0 \supset U_1 \supset \ldots \supset U_i$ is not an $(\mathbf{n}_i, m, \lambda + \varepsilon_i)$-vortex. As $\varepsilon_0 \geq 0$ and $\lambda \geq \mu$, $U_0 = V(G)$ is an $(\mathbf{n}_0, m, \lambda + \varepsilon_0)$-vortex. If no event E_i, $i \in [\ell]$, holds, then $U_0 \supset U_1 \supset \ldots \supset U_\ell$ is an $(\mathbf{n}_0, m, \lambda + \varepsilon_\ell)$-vortex, and hence an $(\mathbf{n}_0, m, 2\lambda)$-vortex as $\varepsilon_\ell = (m/n_\ell)^{1/4} \leq (\gamma_1 K)^{-1/4} \leq \lambda$. Therefore, to prove the claim it is sufficient to show that

$$\sum_{i \in [\ell]} \mathbb{P}(E_i) < \frac{1}{4}.$$

Let $i \in [\ell]$, and suppose that $U_0 \supset U_1 \supset \ldots \supset U_{i-1}$ is an $(\mathbf{n}_{i-1}, m, \lambda + \varepsilon_{i-1})$-vortex. For each $U \subset U_{i-1}$ with $|U| = m$, as $|N(U, U_{i-1})| \geq (1 - \lambda - \varepsilon_{i-1})n_{i-1}$, we have $\mathbb{E}|N(U, U_i)| \geq (1 - \lambda - \varepsilon_{i-1})n_i \geq n_i/2$ and, by Lemma 2.20 with $\varepsilon = (m/n_i)^{1/3}$, that

$$\mathbb{P}\left(|N(U,U_i)| < \left(1 - \lambda - \varepsilon_{i-1} - \left(\frac{m}{n_i}\right)^{1/3}\right)n_i\right) \leq 2\exp\left(-\left(\frac{m}{n_i}\right)^{2/3} \cdot \frac{n_i}{6}\right) = 2\exp\left(-\frac{m}{6} \cdot \left(\frac{n_i}{m}\right)^{1/3}\right). \tag{5.2}$$

Now, as $n_i \leq \gamma_2 n_{i-1}$, we have

$$\varepsilon_i - \varepsilon_{i-1} = \left(\frac{m}{n_i}\right)^{1/4} - \left(\frac{m}{n_{i-1}}\right)^{1/4} = \left(\frac{m}{n_i}\right)^{1/4}\left(1 - \left(\frac{n_i}{n_{i-1}}\right)^{1/4}\right) \geq \left(\frac{m}{n_i}\right)^{1/4}(1 - \gamma_2^{1/4}) \geq \left(\frac{m}{n_i}\right)^{1/3}, \tag{5.3}$$

as $n_i \geq n_\ell \geq \gamma_1 K m$ and $1/K \ll \gamma_1, \gamma_2$, so that $(1 - \gamma_2^{1/4}) \geq (1/\gamma_1 K)^{1/12} \geq (m/n_i)^{1/12}$. Furthermore, in preparation for taking a union bound over all sets $U \subset U_{i-1}$ with size m, as $\gamma_1 n_{i-1} \leq n_i$, we have

$$\begin{aligned}\binom{n_{i-1}}{m} \cdot 2\exp\left(-\frac{m}{6} \cdot \left(\frac{n_i}{m}\right)^{1/3}\right) &\leq \left(\frac{en_{i-1}}{m}\right)^m \cdot 2\exp\left(-\frac{m}{6} \cdot \left(\frac{n_i}{m}\right)^{1/3}\right)\\ &\leq 2\exp\left(\frac{m}{6} \cdot \left(6\log\left(\frac{en_i}{\gamma_1 m}\right) - \left(\frac{n_i}{m}\right)^{1/3}\right)\right)\\ &\leq 2\exp\left(-\frac{m}{12} \cdot \left(\frac{n_i}{m}\right)^{1/3}\right) \leq \frac{m}{n_i},\end{aligned} \tag{5.4}$$

where the last line of inequalities holds as $1/K \ll \gamma_1, \gamma_2$ and $n_i/m \geq n_\ell/m \geq \gamma_1 K$. Therefore, by (5.2), (5.3) and (5.4), and by taking a union bound, we have that, with probability more than $1 - (m/n_i)$, for each $U \subset U_{i-1}$ with $|U| = m$ we have $|N(U, U_i)| \geq (1 - \lambda - \varepsilon_i)n_i$.

Hence, we have

$$\sum_{i\in[\ell]} \mathbb{P}(E_i) \leq \sum_{i\in[\ell]} \frac{m}{n_i} \leq \sum_{i\in[\ell]} \frac{m}{n_\ell} \cdot \gamma_2^{\ell-i} \leq \frac{1}{1-\gamma_2} \cdot \frac{m}{n_\ell} \leq \frac{2m}{n_\ell} \leq \frac{2}{\gamma_1 K} < \frac{1}{4},$$

as required. $\boxdot$

Proof of Claim 5.6. Let E be the event that $|W| \geq \frac{n_1 - n_2}{6}$. By (5.1), $\mathbb{P}(E) \geq 7/8$. Note that, if E holds, then $|W| \geq \gamma_1 n/8 \geq \mu n + (2\gamma_1 D + 2)m$, as $n_1 - n_2 \geq (1 - \gamma_2)\gamma_1 n \geq 7\gamma_1 n/8$ and $\mu \ll 1/D \ll \gamma_1$. Thus, for every choice of W for which E holds, we can apply Proposition 2.13 to find $B_W \subset V(G)$ such that $|B_W| < m$ and, for each $U \subset V(G) \setminus B_W$ with $|U| \leq m$, we have $|N_G(U, W)| \geq \gamma_1 D|U|$.

Let E' be the event that E holds and $|N_G(U, W')| \geq \gamma_1 Dm/100$ for every $U \subset B_W$ with $|U| = \lfloor m/8 \rfloor$ and $|N_G(U, W)| < \gamma_1 Dm/100$. Note that, conditioned on any choice of W for which E holds, as G is a (D, m)-expander, for every set $U \subset B_W$ with $|U| = \lfloor m/8 \rfloor$ and $|N_G(U, W)| < \gamma_1 Dm/100$, we have $|N_G(U, V(G) \setminus W)| \geq D|U| - \gamma_1 Dm/100 \geq D|U|/2$. Thus, by Lemma 2.20 with $\varepsilon = 1/1000$ and a union bound,

$$\mathbb{P}(E' \text{ does not hold}|E \text{ holds}) \leq \binom{m}{\lfloor m/8 \rfloor} \cdot 2\exp\left(\frac{-\gamma_1 Dm}{6 \cdot 10^8}\right) < \frac{1}{8},$$

as $1/m, 1/D \ll \gamma_1$. Therefore, $\mathbb{P}(E' \text{ holds}) \geq \mathbb{P}(E) - \mathbb{P}(E' \text{ does not hold}|E \text{ holds}) > 3/4$.

We claim that if E' holds, then, for every subset $U \subset V(G)$ with $|U| = \lfloor m/4 \rfloor$, $|N_G(U, W \cup W')| \geq \gamma_1 Dm/200$. Indeed, for a set $U \subset V(G)$ with $|U| = \lfloor m/4 \rfloor$, if $|U \cap B_W| \geq \lfloor m/8 \rfloor$, then either $|N_G(U, W)| \geq \gamma_1 Dm/100 - |U| \geq \gamma_1 Dm/200$, or, as E' holds, $|N_G(U, W')| \geq \gamma_1 Dm/100 - |U| \geq \gamma_1 Dm/200$. On the other hand, if $|U \cap B_W| < \lfloor m/8 \rfloor$, then $|U \setminus B_W| \geq \lfloor m/8 \rfloor$, and so by the choice of B_W, $|N_G(U) \cap W| \geq \gamma_1 Dm/8 - |U| \geq \gamma_1 Dm/200$, as required. Since E' holds with probability in excess of 3/4, the claim follows. $\boxdot$

Proof of Claim 5.7. We first show that $G[V_0]$ is not $(m, \lambda m)$-joined with probability less than $1/12$. Note that, for each $U \subset V(G)$ with $|U| = m$ we have $|V(G) \setminus (U \cup N(U))| \leq \mu n$, and therefore

$$\mathbb{P}(U \subset V_0 \text{ and } |V_0 \setminus (U \cup N(U))| > \lambda m) \leq p_0^m \cdot \binom{\mu n}{\lambda m} p_0^{\lambda m}. \tag{5.5}$$

Thus, the probability that $G[V_0]$ is not $(m, \lambda m)$-joined is, as $p_0 = 2n_k/n_0 = 2n_k/n$, at most

$$\begin{aligned}\binom{n}{m} p_0^m \cdot \binom{\mu n}{\lambda m} p_0^{\lambda m} &\leq \left(\frac{ep_0 n}{m}\right)^m \cdot \left(\frac{ep_0 \mu n}{\lambda m}\right)^{\lambda m} = \left(\frac{2en_k}{m}\right)^m \cdot \left(\frac{2e\mu n_k}{\lambda m}\right)^{\lambda m} \leq (2eK_0)^m \cdot \left(\frac{2e\mu K_0}{\lambda}\right)^{\lambda m} \\ &= (2eK_0)^{(1+\lambda)m} \cdot \left(\frac{\mu}{\lambda}\right)^{\lambda m} \leq K_0^{3m} \cdot \left(\frac{\mu}{\lambda}\right)^{\lambda m} \leq \frac{1}{12},\end{aligned} \tag{5.6}$$

where the last inequality follows as $\mu \ll 1/K_0 \ll \lambda$.

From the proof of Claim 5.4, we have $\mathbb{P}(|V_0| > 4n_k) < 1/12$. Furthermore, if $G[V_0]$ is $(m, \lambda m)$-joined and $|V_0| \leq 4n_k$, then, by similar calculations to those in (5.5) and (5.6) and as $p_1 = 2n_{\ell-1}/n_k$, $n_{\ell-1} \leq 2Km/\gamma_1$ and $n_k \geq \gamma_1 K_0 m$, we have that $G[V_1]$ is not (λm)-joined with probability at most

$$\binom{4n_k}{\lambda m} p_1^{\lambda m} \cdot \binom{m}{\lambda m} p_1^{\lambda m} \leq \left(\frac{4ep_1 n_k}{\lambda m}\right)^{\lambda m} \cdot \left(\frac{ep_1 m}{\lambda m}\right)^{\lambda m} = \left(\frac{16e^2 n_{\ell-1}^2}{\lambda^2 m n_k}\right)^{\lambda m} \leq \left(\frac{64e^2 K^2}{\lambda^2 \gamma_1^3 K_0}\right)^{\lambda m} < \frac{1}{12},$$

where the last inequality holds as $1/K_0 \ll 1/K \ll \lambda, \gamma_1$ and $1/m \ll \lambda$.

Altogether then, with probability in excess of $3/4$, $G[V_1]$ is (λm)-joined. $\boxdot$ $\square$

5.2 Vortex partition

We now use Lemma 5.3 to find a vortex $U_0 \supset U_1 \supset \ldots \supset U_\ell$ and then run a simple 'cleaning procedure' to find disjoint sets $V_1, V_2, \ldots, V_\ell$ for a vortex partition (see Definition 5.1) such that V_i is a subset of $U_i \setminus U_{i+1}$ which contains almost all those vertices for each $i \in [\ell]$, before then completing the vortex partition by using all but at most $m/4$ of the remaining vertices not in $\cup_{i\in[\ell]} V_i$ to form V_0. This will give us the following result.

Lemma 5.8. *Let*

$$\frac{1}{m} \ll \mu \ll \frac{1}{K}, \frac{1}{D} \ll \lambda \ll \frac{1}{d} \ll \gamma_1, \gamma_2 \leq \frac{1}{16},$$

with $\gamma_1 < \gamma_2$ and let n be such that $m \leq \mu n$. Let G be an n-vertex $(m, \mu n)$-joined graph which is a (D, m)-expander. Let $\ell \in \mathbb{N}$ and let $\mathbf{n} = (n_0, n_1, \ldots, n_\ell)$ be a (γ_1, γ_2)-descending tuple with $n = \sum_{i\in[\ell]_0} n_i$ and $\gamma_1 Km \leq n_\ell \leq 2Km$.

Then, G contains a subgraph G' with at least $n - \lfloor m/4 \rfloor$ vertices that has an $(\mathbf{n}, 2\lambda, d)$-vortex partition $V_0 \cup V_1 \cup \ldots \cup V_\ell$ such that $G[V_{\ell-1} \cup V_\ell]$ is (λm)-joined.

Proof. Let $\mathbf{n}' = (n_0', n_1', \ldots, n_\ell')$ with $n_i' = \sum_{j=i}^{\ell} n_i$ for each $i \in [\ell]_0$. Note that $\mathbf{n}'$ is $(\gamma_1/2, \gamma_2)$-descending, $n_j'/2 \leq n_j \leq n_j'$ for all $j \in [\ell]_0$, and $\gamma_1 Km \leq n_\ell' \leq 2Km$. Therefore, by Lemma 5.3, G contains an $(\mathbf{n}', m, 2\lambda)$-vortex $U_0 \supset U_1 \supset \ldots \supset U_\ell$ such that $G[U_{\ell-1}]$ is (λm)-joined and, for every set $U \subset V(G)$ with $|U| = \lfloor \frac{m}{4} \rfloor$, $|N_G(U, U_1 \setminus U_2)| \geq \gamma_1 Dm/200$. For each $i \in [\ell - 1]_0$, let $V_i' = U_i \setminus U_{i+1}$, and let $V_\ell' = U_\ell$. Note that, for each $i \in [\ell]_0$, $|V_i'| = n_i$.

Let $W_{\ell+1} = \emptyset$. Now, for each $j = \ell, \ell - 1, \ldots, 1$ in turn, do the following, where W_{j+1} is a set chosen (if $j < \ell$) in the previous iteration such that $|W_{j+1}| < m$.

G1 Note that, as $\lambda \ll \gamma_1$ and $d \ll K$, we have $|V_j' \setminus W_{j+1}| \geq n_j - m \geq 2\lambda n_{j-1} + (2d + 2)m$.

G2 Note that, by **F2** and $V_{j-1}' \cup V_j' \subset U_{j-1}$, we have that $G[V_{j-1}' \cup (V_j' \setminus W_{j+1})]$ is $(m, \lambda n_{j-1}')$-joined, and hence $(m, 2\lambda n_{j-1})$-joined.

G3 Using Proposition 2.13, take $W_j \subset V'_{j-1} \cup (V'_j \setminus W_{j+1})$ such that $|W_j| < m$ and, for each $U \subset (V'_{j-1} \cup V'_j) \setminus (W_j \cup W_{j+1})$ with $|U| \leq m$, $|N_G(U, V'_j \setminus (W_j \cup W_{j+1}))| \geq d|U|$.

G4 Let $V_j = V'_j \setminus (W_j \cup W_{j+1})$.

Now, for every set $U \subset V(G)$ with $|U| = \lfloor m/4 \rfloor$, we have

$$|N_G(U, V_1)| \geq |N_G(U, V'_1)| - |W_1 \cup W_2| \geq \gamma_1 Dm/200 - 2m \geq \gamma_1 Dm/500,$$

as $1/D \ll \gamma_1$. Let $W_0 \subset V(G)$ be a maximal subset subject to $|W_0| < m/2$ and $|N_G(W_0, V_1)| \leq 10d|W_0|$. Similarly to the proof of Proposition 2.13 and using $d \ll \gamma_1 D$, we have $|W_0| < m/4$, and, for each $U \subset V(G) \setminus W_0$ with $|U| \leq m/4$, $|N_G(U, V_1)| \geq 10d|U|$. Let $V_0 = V(G) \setminus (W_0 \cup V_1 \cup \ldots \cup V_\ell)$ and $G' = G[V_0 \cup V_1 \cup \ldots \cup V_\ell]$.

Note that $|G'| \geq n - m/4$, and that, as $V_{\ell-1} \cup V_\ell \subset V'_{\ell-1} \cup V'_\ell = U_{\ell-1}$, $G[V_{\ell-1} \cup V_\ell]$ is (λm)-joined. Therefore, to complete the proof it is left only to show that $V_0 \cup V_1 \cup \cdots \cup V_\ell$ is an $(\mathbf{n}, 2\lambda, d)$-vortex partition of G'.

Since $V'_0 \setminus W_0 \subset V_0 \subset V'_0 \cup W_1 \cup \ldots \cup W_\ell$, $|W_0| \leq \lfloor m/4 \rfloor$ and $|W_1 \cup \ldots \cup W_\ell| \leq \ell m$, we have $(1-2\lambda)n_0 \leq |V_0| \leq (1+2\lambda)n_0$, proving the $j = 0$ case of **E1**. For all $j \in [\ell]$, $|V'_j| = n_j$, $V_j = V'_j \setminus (W_j \cup W_{j+1})$ and $|W_j|, |W_{j+1}| < m \leq \lambda n_j$, so we have $|V_j| \geq n_j - 2m \geq (1-2\lambda)n_j$ and **E1** holds. Now let $2 \leq j \leq \ell$. Since $V_{j-1} \subset V'_{j-1}$ and $V_j \subset V'_j \setminus W_{j+1}$, $G[V_{j-1} \cup V_j]$ is $(m, 2\lambda n_{j-1})$-joined by **G2**, hence $(2\lambda n_{j-1})$-joined, and thus **E2** holds. For **E3**, let $U \subset V_{j-1} \cup V_j \subset (V'_{j-1} \cup V'_j) \setminus (W_j \cup W_{j+1})$ satisfy $|U| \leq 4\lambda n_{j-1}$. If $|U| \leq m$, we have $|N(U, V_j)| \geq d|U|$ by **G3**. If $|U| > m$, then since $G[V_{j-1} \cup V_j]$ is $(m, 2\lambda n_{j-1})$-joined and $d\lambda \ll \gamma_1$, we have

$$|N(U, V_j)| \geq |V_j| - |U| - 2\lambda n_{j-1} \geq (n_j - 2m) - 4\lambda n_{j-1} - 2\lambda n_{j-1} \geq 4d\lambda n_{j-1} \geq d|U|.$$

Hence, $I(V_{j-1})$ is $(d, 2\lambda n_{j-1})$-extendable in $G[V_{j-1} \cup V_j]$ by Proposition 2.15, and so **E3** holds.

For the $j = 1$ case, **E2** holds as $G \supset G[V_0 \cup V_1]$ is $(m, \mu n)$-joined, thus $(m, 2\lambda n_0)$-joined and $(2\lambda n_0)$-joined. Let $U \subset V_0 \cup V_1$ satisfy $|U| \leq 4\lambda n_0$. If $|U| \leq \lfloor m/4 \rfloor$, then $|N(U, V_1)| \geq 10d|U| \geq d|U|$. If $\lfloor m/4 \rfloor < |U| \leq m$, then let $U' \subset U$ have size $\lfloor m/4 \rfloor$, and thus

$$|N(U, V_1)| \geq |N(U', V_1)| - |U \setminus U'| \geq 10d|U'| - |U| \geq 2dm - m \geq dm \geq d|U|.$$

If $|U| > m$, then, since $G[V_0 \cup V_1]$ is $(m, 2\lambda n_0)$-joined, we have

$$|N(U, V_1)| \geq |V_1| - |U| - 2\lambda n_0 \geq n_1 - 2m - 6\lambda n_0 \geq 4d\lambda n_0 \geq d|U|.$$

Thus, we have that $I(V_0)$ is $(d, 2\lambda n_0)$-extendable in $G[V_0 \cup V_1]$, proving the $j = 1$ case of **E3**. This completes the proof that $V_0 \cup V_1 \cup \ldots \cup V_\ell$ is an $(\mathbf{n}, d, 2\lambda)$-vortex partition of G'. □

5.3 Proof of Theorem 2.5

Using Lemma 5.8, we can now prove Theorem 2.5 following the outline at the start of this section.

Proof of Theorem 2.5. Let $d = 10^4\Delta^5$, $\gamma = 1/10\Delta$ and note that $d \gg \Delta^3/\gamma$. Let

$$\frac{1}{m} \ll \mu \ll \frac{1}{K}, \frac{1}{D} \ll \lambda \ll \frac{1}{d}.$$

Let $\gamma_1 = \gamma/4\Delta$ and $\gamma_2 = 2\gamma$, and let T be a tree with $n - m + 1$ vertices and $\Delta(T) \leq \Delta$. Using Lemma 2.11, we can find some $\ell \in \mathbb{N}$, some $\mathbf{n}' = (n'_1, \ldots, n'_\ell) \in \mathbb{N}^\ell$ that is (γ_1, γ_2)-descending with $\gamma Km/3\Delta \leq n'_\ell \leq Km$, and an $\mathbf{n}'$-decomposition $(T_1, \ldots, T_\ell)$ of T. For $1 \leq i \leq \ell - 1$, let t_i be the leaf of T_i in T_{i+1}, and let t_ℓ be an arbitrary leaf of T_ℓ which is not $t_{\ell-1}$. As every tree with at least 2 vertices has at least 2 leaves, we may also take a leaf t_0 of T_1 which is not t_1.

Let $n_0 = \lceil (1-\gamma_2)n'_1 \rceil$, let $n_j = \lfloor \gamma_2 n'_j \rfloor + \lceil (1-\gamma_2)n'_{j+1} \rceil$ for each $j \in [\ell-1]$, and let $n_\ell = \lfloor \gamma_2 n'_\ell \rfloor + m - 1$. Note that, for each $j \in [\ell]$, $n_j \leq n'_j$ and

$$\sum_{i=j}^{\ell} n_i = \lfloor \gamma_2 n'_j \rfloor + \sum_{i=j+1}^{\ell} n'_i + (m-1), \tag{5.7}$$

and

$$\sum_{i=0}^{\ell} n_i = \sum_{i=1}^{\ell} n'_i + (m-1) = |T| + (m-1) = n. \tag{5.8}$$

Moreover, using $\mathbf{n}'$ is (γ_1, γ_2)-descending, we can verify that $\mathbf{n} = (n_0, n_1, \ldots, n_\ell)$ is $(\gamma_1/2, 3\gamma_2)$-descending. Let $K' = K/5\Delta$ and note that $n_\ell = \lfloor \gamma_2 n'_\ell \rfloor + m - 1 \leq 2\gamma_2 n'_\ell \leq 2K\gamma_2 m = 2Km/5\Delta = 2K'm$ as $\gamma_2 = 1/5\Delta$, while $n_\ell \geq \lfloor \gamma_2 n'_\ell \rfloor \geq \gamma_2 \cdot \gamma K m/6\Delta \geq \gamma_2 \cdot 5\gamma K' m/6 = \gamma K' m/6\Delta \geq \gamma_1 K' m/2$ as $\gamma_1 = \gamma/4\Delta$. Since $\mu \ll 1/K' \ll \lambda$, using Lemma 5.8, we can take a subgraph G' of G with at least $n - \lfloor m/4 \rfloor$ vertices which has an $(\mathbf{n}, 2\lambda, d)$-vortex partition, $V_0 \cup V_1 \cup \ldots \cup V_\ell$ say, such that $G[V_{\ell-1} \cup V_\ell]$ is (λm)-joined.

For each $j \in [\ell]_0$, say that we have a *stage j situation* if we have distinct vertices $v_0, v_1, \ldots, v_j$ and copies $S_1, \ldots, S_j$ of the trees $T_1, \ldots, T_j$, respectively, so that the following hold.

H1 If $i \in [j]_0$, then $v_i \in V_i$.

H2 If $i \in [j]$, then $\cup_{i'=0}^{i-1} V_{i'} \subset \cup_{i'=1}^{i} V(S_{i'}) \subset \cup_{i'=0}^{i} V_{i'}$.

H3 For each $i \in [j]$, t_{i-1} is copied to v_{i-1} in S_i and t_i is copied to v_i in S_i.

H4 For each $1 \leq i < i' \leq j$, $V(S_i) \cap V(S_{i'}) = \{v_i\} \cap \{v_{i'-1}\}$.

Arbitrarily, pick $v_0 \in V_0$, and note that this gives us a stage 0 situation as **H1** holds and **H2**–**H4** are vacuous. Furthermore, if we have a stage ℓ situation, with vertices $v_0, v_1, \ldots, v_\ell$ and copies $S_1, \ldots, S_\ell$ of the trees $T_1, \ldots, T_\ell$, respectively, then **H3** and **H4** imply that $\cup_{i \in [\ell]} S_i$ is a copy of $T = \cup_{i \in [\ell]} T_i$ in G', and hence G has a copy of T, as required. Thus, the lemma is implied by the following claim and induction.

Claim 5.9. *For each $j \in [\ell]$, if we have a stage $j-1$ situation, then we can create a stage j situation.*

Proof of Claim 5.9. Set S_0 to be the tree containing only the vertex v_0. Fix $j \in [\ell]$ and let $X_j = V_{j-1} \setminus V(S_{j-1})$. Now, as $X_j \cup \{v_{j-1}\} \subset V_{j-1}$, and $V_0 \cup V_1 \cup \ldots \cup V_\ell$ is an $(\mathbf{n}, 2\lambda, d)$-vortex partition, we have the following by **E3**.

I $I(X_j \cup \{v_{j-1}\})$ is $(d, \lambda n_{j-1})$-extendable in $G[X_j \cup \{v_{j-1}\} \cup V_j]$.

Then, if $j \leq \ell - 1$,

$$\begin{aligned}
|X_j \cup \{v_{j-1}\} \cup V_j| &= 1 + |(V_{j-1} \cup V_j) \setminus V(S_{j-1})| \overset{\mathbf{H2}}{=} 1 + \left| \left(\cup_{i=0}^{j} V_i\right) \setminus \left(\cup_{i=0}^{j-1} V(S_i)\right) \right| \\
&= 1 + |G'| - |\cup_{i=j+1}^{\ell} V_i| - \left(|T| - \sum_{i=j}^{\ell} n'_i\right) \\
&\overset{\mathbf{E1}}{\geq} 1 + \left(n - \left\lfloor \frac{m}{4} \right\rfloor\right) - \sum_{i=j+1}^{\ell} (1+\lambda) n_i - |T| + \sum_{i=j}^{\ell} n'_i \\
&\overset{(5.7)}{\geq} \left(n + 1 - |T| - \left\lfloor \frac{m}{4} \right\rfloor\right) - \sum_{i=j+1}^{\ell} n_i - 2\lambda n_{j+1} + \sum_{i=j}^{\ell} n'_i \\
&\overset{(5.7)}{\geq} \left(m - \left\lfloor \frac{m}{4} \right\rfloor\right) - \left(\lfloor \gamma_2 n'_{j+1} \rfloor + \sum_{i=j+2}^{\ell} n'_i + m - 1\right) - 2\lambda n'_{j+1} + \sum_{i=j}^{\ell} n'_i
\end{aligned}$$

$$
\begin{aligned}
&\geq n'_j + \frac{n'_{j+1}}{2} - \frac{m}{2} \geq \left(1 + \frac{\gamma_1}{2}\right) n'_j - \frac{m}{2} \\
&\geq \left(1 + \frac{\gamma_1}{2}\right) |T_j| - m \geq |T_j| + 20dm, \qquad (5.9)
\end{aligned}
$$

where the last inequality follows from $\gamma_1|T_j| \geq \gamma_1 \cdot \gamma K m/3\Delta \geq 20dm$. Furthermore,

$$
\begin{aligned}
|X_j| = |V_{j-1} \setminus V(S_{j-1})| &\overset{\mathbf{H2}}{=} \left|\left(\cup_{i=0}^{j-1} V_i\right) \setminus \left(\cup_{i=0}^{j-1} V(S_i)\right)\right| \\
&= |G'| - |\cup_{i=j}^{\ell} V_i| - (|T| - \sum_{i=j}^{\ell} n'_i) \\
&\leq n - \sum_{i=j}^{\ell} (1-\lambda) n_i - |T| + \sum_{i=j}^{\ell} n'_i \\
&\leq m + 2\lambda n_j - \sum_{i=j}^{\ell} n_i + \sum_{i=j}^{\ell} n'_i \\
&\leq n'_j + 2\lambda n'_j - \lfloor \gamma_2 n'_j \rfloor + 1 \leq \left(1 - \frac{\gamma_2}{2}\right) n'_j \\
&\leq \left(1 - \frac{\gamma_2}{2}\right) |T_j| = (1-\gamma)|T_j|. \qquad (5.10)
\end{aligned}
$$

Finally, we have $|T_j| \geq n'_j \geq \gamma_1 n'_{j-1} \geq \gamma_1 n_{j-1} \geq 2d^2 \lambda n_{j-1}$ and $G[X_j \cup \{v_{j-1}\} \cup V_j]$ is (λn_{j-1})-joined by **E2**. Therefore, by **I**, (5.9), (5.10) and Lemma 4.1, we can find a copy S_j of T_j in $G[X_j \cup \{v_{j-1}\} \cup V_j]$ with t_{j-1} copied to v_{j-1}, t_j copied into V_j, and $X_j \subset V(S_j)$. Then, letting v_j to be the copy of t_j, we have a stage j situation.

Suppose then that $j = \ell$. From **I**, we have that $I(X_\ell \cup \{v_{\ell-1}\})$ is $(d, \lambda n_{\ell-1})$-extendable in $G[X_\ell \cup \{v_{\ell-1}\} \cup V_\ell]$, and hence $(d, \lambda m)$-extendable. As $X_\ell \cup \{v_{\ell-1}\} \cup V_\ell \subset V_{\ell-1} \cup V_\ell$, we have that $G[X_\ell \cup \{v_{\ell-1}\} \cup V_\ell]$ is (λm)-joined. Furthermore,

$$
\begin{aligned}
|X_\ell \cup \{v_{\ell-1}\} \cup V_\ell| = 1 + |(V_{\ell-1} \cup V_\ell) \setminus V(S_{\ell-1})| &\overset{\mathbf{H2}}{=} 1 + \left|\left(\cup_{i=0}^{\ell} V_i\right) \setminus \left(\cup_{i=0}^{\ell-1} V(S_i)\right)\right| \\
&\geq 1 + |G'| - (|T| - |T_\ell| + 1) \geq n - \left\lfloor \frac{m}{4} \right\rfloor - |T| + |T_\ell|. \\
&\geq |T_\ell| + \frac{m}{2} \geq |T_\ell| + 10\lambda dm.
\end{aligned}
$$

Therefore, by Corollary 2.19, there is a copy S_ℓ of T_ℓ in $G[X_\ell \cup \{v_{\ell-1}\} \cup V_\ell]$ in which $t_{\ell-1}$ is copied to $v_{\ell-1}$. Let v_ℓ be the copy of t_ℓ and note that we have a stage ℓ situation, as required. This completes the proof of the claim and hence the theorem. ⊡ □

6 Ramsey goodness of bounded degree trees

In this section, we prove Theorem 2.1 by induction from Theorem 2.2. To embed a tree T in a graph G in the setting of Theorem 2.1, we will again show we can assume some extra expansion condition, before dividing into 3 (overlapping) cases. Roughly, these cases are the following.

a) T has linearly many leaves.

b) G is not well connected.

c) G is well connected and T does not have linearly many leaves.

In each case the embedding is different, and we sketch these at the start of Sections 6.1–6.3 respectively, before combining them to prove Theorem 2.1 in Section 6.4.

6.1 Trees with linearly many leaves

We first show that, in the setting of Theorem 2.1, if T has linearly many leaves and G satisfies a simple expansion condition (**J**), then G contains a copy of T. This was shown by Balla, Pokrovskiy, and Sudakov [2], but for completion we include the full proof, following their method while using the extendability framework.

To embed T, we remove a large set of leaves, and embed the resulting tree into the graph in an extendable fashion. We then show that the relevant extendability conditions imply that a version of Hall's matching criterion holds in the appropriate subgraph to allow unused vertices to be attached to the embedded tree as leaves in order to create a copy of the original tree.

Lemma 6.1. *Let $1/n \ll \mu \ll 1/\Delta$. Let G be a graph with at least n vertices in which the following holds.*

J *Every set $U \subset V(G)$ with $|U| = \mu n$ satisfies $|U \cup N_G(U)| \geq n$.*

Then, G contains a copy of every n-vertex tree T with $\Delta(T) \leq \Delta$ and at least $10\Delta^2\mu n$ leaves.

Proof. Let T be an n-vertex tree with $\Delta(T) \leq \Delta$ and at least $10\Delta^2\mu n$ leaves. Let $d = 2\Delta$, $m = \mu n$ and $n_0 = |G| - n + 1$, so that G is (m, n_0)-joined by **J**, and

$$|G| = n_0 + n - 1 \geq n_0 + (1 - 10\Delta\mu)n + (4d + 2)m. \tag{6.1}$$

Using Proposition 2.6, we can find $W \subset V(G)$ with $|W| < m$ such that $G' = G - W$ is a $(2d, m)$-expander.

Arbitrarily, pick $v \in V(G')$. Note that, for each $U \subset V(G')$ with $1 \leq |U| \leq m$, as G' is a $(2d, m)$-expander, we have $|N_{G'}(U)| \geq 2d|U| \geq 1 + d|U|$. By **J**, if $U \subset V(G')$ with $m \leq |U| \leq 2m$, we also have $|N_{G'}(U)| \geq n - 2m \geq 1 + 2dm \geq 1 + d|U|$. Thus, by Proposition 2.15, $I(\{v\})$ is (d, m)-extendable in G'.

As T has at least $10\Delta^2\mu n$ leaves, we can find a set of leaves L such that $|L| = 10\Delta\mu n$ and no pair of leaves in L has a common neighbour in T. Letting $T' = T - L$, we will now embed T' into G'. Note that

$$1 + |T'| \leq 1 + n - 10\Delta\mu n \overset{(6.1)}{\leq} |G'| - (2d + 2)m - n_0.$$

Then, as G, and thus G', is (m, n_0)-joined, by Corollary 2.19 (applied with an arbitrary $t \in V(T')$), there is a (d, m)-extendable subgraph S' of G' which is a copy of T'.

Let $A \subset V(S')$ be the copy in S' of the set of parents in T of the leaves in L, and let $B = V(G) \setminus V(S')$. Observe that to make S' into a copy of T, it is sufficient to find a matching in G from A to B and add this to S'. We will find such a matching by showing that the appropriate Hall's matching criterion holds.

If $U \subset A$, with $|U| \leq m$, then, as S' is (d, m)-extendable in G', by (2.1) we have

$$|N_G(U, B)| \geq |N_{G'}(U) \setminus V(S')| = |N'_{G'}(U) \setminus V(S')| \geq (d - 1)|U| - (\Delta - 1)|U| \geq |U|.$$

On the other hand, if $U \subset A$ with $|U| \geq m$, then

$$|N_G(U, B)| \geq |U \cup N_G(U)| - |T'| \overset{\mathbf{J}}{\geq} n - (n - |L|) = |A| \geq |U|.$$

Therefore, for each $U \subset A$ we have $|N_G(U, B)| \geq |U|$, and thus, as Hall's matching criterion holds, G has a matching from A to B, as required. □

6.2 Non-well-connected graphs

For the next case in the induction step of Theorem 2.1, which we prove as Lemma 6.2 below, we will use properties that will follow from induction (see **K1** below) and our graph G will not be well-connected, in the sense that it will have a vertex partition $V_0 \cup V_1 \cup V_2$ in which V_0 contains at most a small linear (in n) number of vertices and there are no edges in G between V_1 and V_2, which are both large sets (see **K2**). We will start by, for each $i \in [2]$, finding a large subset $V_i' \subset V_i$, so that $G[V_i']$ has an expansion condition, for otherwise we will be done by the induction properties. Neither of these properties will be necessarily strong enough to embed T. However, if a vertex has Δ neighbours in both V_1' and V_2' (this is Case I in the proof of Lemma 6.2), then this will allow us to embed part of the tree in $G[V_1']$ and part of the tree in $G[V_2']$ in such a way that they can be connected through this vertex to get a copy of T. Thus, we can then assume that no such vertex exists (Case II in the proof of Lemma 6.2). This will allow us to partition $V(G) \setminus (V_1' \cup V_2')$ as $U_1 \cup U_2$ so that there are very few edges between V_1' and U_2 and very few edges between V_2' and U_1. Assuming that G contains no copy of T, we then apply induction to find complete partite graphs in $G^c[V_1' \cup U_1]$ and $G^c[V_2' \cup U_2]$ which we can combine (with the deletion of some vertices from the larger parts) to get a complete k-partite graph in G^c with the class sizes we need.

Lemma 6.2. *Let $k \geq 3$ and $1/n \ll \lambda \ll \mu \ll 1/\Delta$. Let $m \leq \lambda n$ and let G be a graph with $(k-1)(n-1)+m$ vertices. Let T be an n-vertex tree with $\Delta(T) \leq \Delta$. Suppose the following hold.*

K1 *For any $2 \leq k' \leq k-1$ and $m' \leq \mu n$, $R(T, K_{\mu n}^{k'-1} \times K_{m'}^c) = (k'-1)(n-1)+m'$.*

K2 *There is a partition $V(G) = V_0 \cup V_1 \cup V_2$ such that $e_G(V_1, V_2) = 0$, $|V_1|, |V_2| \geq m$ and $|V_0| \leq \lambda n$.*

Then, G contains a copy of T or G^c contains a copy of $K_{\lambda n}^{k-1} \times K_m^c$.

Proof. Assume for a contradiction that G contains no copy of T and G^c contains no copy of $K_{\lambda n}^{k-1} \times K_m^c$. We start by showing that we have the following property.

K3 For every $U \subset V(G)$ with $|U| = m$, we have $|N_G(U)| \geq (1-2\lambda)n$.

Indeed, if there is some set $U \subset V(G)$ with $|U| = m$ with $|N_G(U)| < (1-2\lambda)n$, then $|G - N_G(U) - U| > (k-2)(n-1) + \lambda n$, and thus, by **K1**, as $G - N_G(U) - U$ contains no copy of T, its complement contains a copy of $K_{\lambda n}^{k-1}$. Adding U as a vertex class, we obtain a copy of $K_{\lambda n}^{k-1} \times K_m^c$ in G^c, contradiction. Thus, **K3** holds.

Now, let $d = 4\Delta$ and consider the partition $V(G) = V_0 \cup V_1 \cup V_2$ given by **K2**. For each $i \in [2]$ and any $U \subset V_i$ with size m, we have $N_G(U) \subset V_i \cup V_0$, so that, by **K3**, $|N_G(U, V_i)| \geq (1-2\lambda)n - |V_0| \geq (1-3\lambda)n$. In particular, $|V_i| \geq m + (1-3\lambda)n$. For each $i \in [2]$, then, letting $n_i = |V_i| - (1-3\lambda)n$, we have that $G[V_i]$ is (m, n_i)-joined and, as $\lambda \ll 1/\Delta$ and $m \leq \lambda n$,

$$|V_i| = n_i + (1-3\lambda)n \geq n_i + \left(1 - \frac{1}{4\Delta}\right)n + 1 + (4d+2)m + 2m. \tag{6.2}$$

In particular, $|V_i| \geq n_i + (4d+2)m$, so, by Proposition 2.6, there is some $W_i \subset V_i$ with $|W_i| < m$ such that $G[V_i] - W_i$ is a $(2d, m)$-expander. For each $i \in [2]$, let $V_i' = V_i \setminus W_i$.

We now consider 2 cases, depending on whether there is some $v \in V(G) \setminus (V_1' \cup V_2')$ with at least Δ neighbours both in V_1' and in V_2' (Case I) or not (Case II). In Case I, we will show that G contains a copy of T, and in Case II, we will show that G^c contains a copy of $K_{\lambda n}^{k-1} \times K_m^c$, a contradiction either way.

Case I. Suppose then that there is some vertex $v \in V(G) \setminus (V_1' \cup V_2')$ with at least Δ neighbours both in V_1' and in V_2'. Using Proposition 2.10, we can find two subtrees T_1 and T_2 of T that share exactly one vertex t that is a leaf in T_1, such that $E(T) = E(T_1) \cup E(T_2)$ and $n/4\Delta \leq |T_2| \leq n/2$. Let t_0 be the unique vertex in T_1 adjacent to t and let $t_1, \ldots, t_r$ be the neighbours of t in T_2. For each $i \in [r]$, let T_i' be the tree in $T_2 - t$ containing t_i. Using that v has at least $\Delta \geq r$ neighbours in V_1' and V_2', we can find a neighbour v_0 of v in V_1' and r distinct neighbours $v_1, \ldots, v_r$ of v in V_2'.

Now, as $G[V_1] - W_1 = G[V_1']$ is an (m, n_1)-joined $(2d, m)$-expander, we have that $I(\{v_0\})$ is (d, m)-extendable in $G[V_1']$. Futhermore, as $|T_1| = n - |T_2| + 1 \leq (1 - 1/4\Delta)n + 1$, by (6.2), we have $|V_1'| \geq |T_1| + (2d+2)m + n_1$. Thus, by Corollary 2.19 there is a copy S_1 of $T_1 - t$ in $G[V_1']$ in which t_0 is copied to v_0.

Similarly, as $G[V_2] - W_1 = G[V_2']$ is an (m, n_2)-joined $(2d, m)$-expander, we have that $I(\{v_1, \ldots, v_r\})$ is (d, m)-extendable in $G[V_2']$ as $r \leq \Delta = d/4$. Futhermore, as $\sum_{i \in [r]} |T_i'| = |T_2| - 1 \leq n/2$, by (6.2), we have $|V_2'| \geq r + \sum_{i \in [r]} |T_i'| + (2d+2)m + n_2$. Thus, by induction and Corollary 2.19, for each $j \in [r]$, there are vertex-disjoint copies $S'_1,\dots,S'_j$ of $T'_1,\dots,T'_j$, respectively, such that t_i is copied to v_i for each $i\in[j]$, and $I(\{v_1,\dots,v_r\})\cup(\cup_{i\in[j]}S_i)$ is (d,m)-extendable in $G[V'_2]$. Note that at the end of the induction, $S_2=(\cup_{i=1}^r S'_i)$ is a copy of T_2-t in $G[V'_2]$. Hence, $S_1\cup S_2$ together with the vertex v and edges $vv_0,\dots,vv_r$ form a copy of T in G, contradiction.

Case II. Suppose, then, that we can partition $V(G)\setminus(V'_1\cup V'_2)$ as $U_1\cup U_2$ so that every vertex in U_1 has fewer than Δ neighbours in V'_2 and every vertex in U_2 has fewer than Δ neighbours in V'_1. For each $i\in[2]$, let k_i and m_i be such that $|V'_i\cup U_i|=k_i(n-1)+m_i$ and $0\le m_i<n-1$. Switching the labels if necessary, we can assume that $m_1\ge m_2$. Now, as $V(G)=V'_1\cup U_1\cup V'_2\cup U_2$ is a partition,

$$(k-1)(n-1)+m=(k_1+k_2)(n-1)+(m_1+m_2),$$

and thus, we have that the following hold.

- If $m_1+m_2<n-1$ then $k-1=k_1+k_2$ and $m=m_1+m_2$.
- If $m_2<m$, then as $(k-1)(n-1)+(m-m_2)=(k_1+k_2)(n-1)+m_1$ and $m_1<(n-1)$, we have $k-1=k_1+k_2$ and $m=m_1+m_2$.
- $k_1+k_2\ge k-2$, so if $k_1+k_2\ne k-2$, then $k-1=k_1+k_2$, and hence $m=m_1+m_2$.

Thus, we either have $m_1\ge(n-1)/2$, $m_2\ge m$ and $k_1+k_2=k-2$ (Case II.1) or $k-1=k_1+k_2$ and $m=m_1+m_2$ (Case II.2). Futhermore, as $|V'_i|\ge|V_i|-m\ge(1-3\lambda)n$ for each $i\in[2]$, we have that $k_1,k_2<k-1$.

Case II.1: $m_1\ge(n-1)/2$, $m_2\ge m$ and $k_1+k_2=k-2$. As $k_1<k-1$ and $m_1\ge n/3$, by **K1**, we have that $V'_1\cup U_1$ contains disjoint sets $Y_1,\dots,Y_{k_1+1}$ with $|Y_i|=\mu n$ for each $i\in[k_1+1]$, and there are no edges in G between any pair Y_i and Y_j for each $1\le i<j\le k_1+1$. Furthermore, as $k_2<k-1$ and $m_2\ge m$, by **K1**, we have that $V'_2\cup U_2$ contains disjoint sets $Z_1,\dots,Z_{k_2+1}$ with $|Z_i|=\mu n$ for each $i\in[k_2]$ and $|Z_{k_2+1}|=m$ so that there are no edges in G between any pair Z_i and Z_j for $1\le i<j\le k_2+1$.

Note that

$$|U_1\cup(N(Z_{k_2+1})\cap V'_1)|\le|V_0|+|W_1|+|W_2|+(\Delta-1)|Z_{k_2+1}|\le\lambda n+2m+\Delta m\le\mu n/2. \tag{6.3}$$

Hence, we can find $Y'_i\subset Y_i\setminus(U_1\cup(N(Z_{k_2+1})\cap V'_1))$ with size λn for each $i\in[k_1+1]$. Similarly, since $|U_2|\le|V_0|+|W_1|+|W_2|\le\lambda n+2m\le\mu n/2$, we can find $Z'_i\subset Z_i\setminus U_2$ having size λn for each $i\in[k_2]$. Let $Z'_{k_2+1}=Z_{k_2+1}$, and observe that there no edges in G between any of the sets

$$Y'_1,Y'_2,\dots,Y'_{k_1+1},Z'_1,Z'_2,\dots,Z'_{k_2+1},$$

and these are $k_1+k_2+1=k-1$ sets with size λn and 1 set of size m, so that G^c contains a copy of $K_{\lambda n}^{k-1}\times K_m^c$, contradiction.

Case II.2: $m_1+m_2=m$ and $k_1+k_2=k-1$. As $k_1<k-1$, by **K1**, we have that $V'_1\cup U_1$ contains disjoint sets $Y_1,\dots,Y_{k_1+1}$ with $|Y_i|=\mu n$ for each $i\in[k_1]$, and $|Y_{k_1+1}|=m_1$, such that there are no edges in G between any pair Y_i and Y_j for each $1\le i<j\le k_1+1$. Similarly, by **K1**, we have that $V'_2\cup U_2$ contains disjoint sets $Z_1,\dots,Z_{k_2+1}$ with $|Z_i|=\mu n$ for each $i\in[k_2]$ and $|Z_{k_2+1}|=m_2$, such that there are no edges in G between any pair Z_i and Z_j for each $1\le i<j\le k_2+1$. For each $i\in[k_1]$, using a similar calculation to (6.3), let $Y'_i\subset Y_i\setminus(U_1\cup(N(Z_{k_2+1})\cap V'_1))$ have size λn and let $Y'_{k_1+1}=Y_{k_1+1}$. For each $i\in[k_2]$, again using a similar calculation to (6.3), let $Z'_i\subset Z_i\setminus(U_2\cup(N(Y_{k_1+1})\cap V'_2))$ have size λn, and let $Z'_{k_2+1}=Z_{k_2+1}$. Then, observe that there no edges in G between any of the sets

$$Y'_1,Y'_2,\dots,Y'_{k_1},Z'_1,Z'_2,\dots,Z'_{k_2},Y'_{k_1+1}\cup Z'_{k_2+1},$$

and these are $k_1+k_2=k-1$ sets with size λn and 1 set of size $m_1+m_2=m$, so that G^c contains a copy of $K_{\lambda n}^{k-1}\times K_m^c$, a contradiction. □

6.3 Trees with few leaves in well-connected graphs

For the last case in the induction step for Theorem 2.1, which we prove as Lemma 6.3, we will have that the graph G is well-connected, in that it has no partition $V(G) = V_0 \cup V_1 \cup V_2$ satisfying **K2** (see **L1**), as well as an additional expansion condition which will follow from our induction (see **L2**), and the tree T will have few leaves. That is, we prove the following lemma.

Lemma 6.3. *Let $k \geq 3$ and $1/n \ll \varepsilon \ll \lambda \ll 1/\Delta, 1/k$. Let $m \leq \varepsilon n$ and let G be a graph with $(k-1)(n-1)+m$ vertices. Let T be an n-vertex tree with fewer than $10\Delta^2\varepsilon n$ leaves satisfying $\Delta(T) \leq \Delta$. Suppose the following properties hold.*

L1 *There is no partition $V(G) = V_0 \cup V_1 \cup V_2$ such that $e_G(V_1, V_2) = 0$, $|V_1|, |V_2| \geq m$ and $|V_0| \leq \lambda n$.*

L2 *For every set $U \subset V(G)$ with $|U| = m$, we have $|N_G(U)| \geq (1-\lambda)n$.*

Then, G contains a copy of T or G^c contains a copy of $K^{k-1}_{\varepsilon n} \times K^c_m$.

The property **L1** will imply that any two disjoint sets U and $U' \subset V(G)$, each with size m, will have many vertex-disjoint short paths between them in G. If we pick a small but linear-sized (in n) random set $Z \subset V(G)$, then for any pair of such sets U and U', plenty of these paths all have internal vertices in Z. This we prove as Proposition 6.5, where we do a little additional work so that the paths we can find all have the same prescribed length.

In Lemma 6.3, we have that $k \geq 3$, so that G contains many more vertices than T. As T has few leaves, most of T will consist of bare paths (paths whose internal vertices have degree 2 in the tree T) of some long, constant, length. We will choose a random partition $V(G) = Z \cup V_0 \cup V_1$, with $|V_0| \geq (1-o(1)) \cdot 2n/5$, $|V_1| \geq 11n/10$ and so that Z has the connection property described above. We then embed the tree T steadily into a large subgraph of $G[V_0]$, but, at any available opportunity, try to save vertices from V_0 by embedding a long bare path of T using many vertices in V_1 and at most a small number of vertices from Z, so that only the last vertex embedded is in V_0 (see Figure 2). We do this so that the embedded tree remains extendable in the large subgraph of $G[V_0]$ that we use. After this embedding (described precisely in **a)**–**c)** of the proof of Lemma 6.3), we analyse it and show that enough of the long bare paths are embedded outside of V_0 so that we do not run out of room within V_0, and thus can successfully embed T. This is shown in Claims 6.7–6.9, but, roughly speaking, if we had $2m$ opportunities to embed a long path in T and did not do so outside of V_0, then let U be the set of vertices to which we could have attached a long bare path to extend the embedding of T, and do the following. We find vertex-disjoint long paths $Q_1, \ldots, Q_{2m}$ in G using vertices in V_1 which have not yet been used in the embedding, and show that under our embedding rules we should have embedded one of the paths Q_i connected to a vertex in U using the connectivity property of Z. To find the paths $Q_1, \ldots, Q_{2m}$, we only need a loose result for the Ramsey numbers of paths versus general graphs as the paths have constant length compared to n, but for convenience we will use the following result.

Theorem 6.4 (Pokrovskiy and Sudakov [23])**.** *Let $k \geq 2$ and let H be a graph with $\chi(H) = k$. Then, for all $n \geq 4|H|$, $R(P_n, H) = (k-1)(n-1) + \sigma(H)$.*

We now prove a result showing that, in the above sketch, the random set Z is likely to have the connectivity property we want.

Proposition 6.5. *Let $k \geq 3$ and $1/n \ll \delta \ll 1/\ell \ll \lambda \ll 1/k$. Let $m \leq \delta n$ and let G be a graph with $(k-1)(n-1)+m$ vertices such that* **L1** *holds and G^c contains no $K^k_{\delta n}$. Let $Z \subset V(G)$ be a set formed by including each element independently at random with probability $1/5$.*

Then, with probability more than $2/3$, for any two disjoint sets $U, U' \subset V(G)$ with size m, there are at least δn internally vertex-disjoint paths with length ℓ through Z connecting U and U' in G.

Proof. First, let $U, U' \subset V(G)$ be disjoint sets and let $Z_0 \subset V(G) \setminus (U \cup U')$ satisfy $|U| = |U'| = m$ and $|Z_0| < \lambda n/2$. We will show that there is a path from U to U' with interior vertices not in $Z_0 \cup U \cup U'$, and with length at least 2 and at most ℓ.

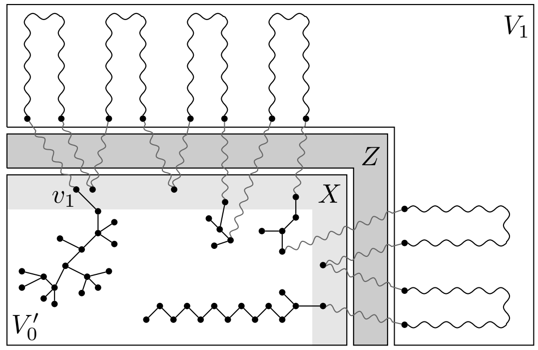

Figure 2: The embedding of T in the proof of Lemma 6.3. Starting with the extendable subgraph $I(X)$ in $G[V_0]$, in which a vertex of T is embedded to v_1, we steadily embed T into G so that many of the long bare paths use vertices in V_1 along with some vertices in Z to connect the path back into $G[V_0]$ (the grey paths depicted have all their interior vertices in Z), and so that the union of $I(X)$ and the subgraph embedded in V_0' remains extendable.

Let $r = \lfloor \ell/2 \rfloor$. Let $U_0 = U$ and, iteratively, for each $1 \leq i \leq r$, let $U_i = U_{i-1} \cup N_{G-Z_0-U'}(U_{i-1})$. Now, the sets $U_i \setminus U_{i-1}$, $i \in [r]$, are disjoint, so there must be some $j \in [r]$ with $|U_j \setminus U_{j-1}| \leq kn/r < \lambda n/2 - m$, as $1/\ell \ll \lambda, 1/k$ and $m \leq \delta n \leq \lambda n/4$. Then, $|Z_0 \cup U' \cup (U_j \setminus U_{j-1})| < \lambda n$, $|U_{j-1}| \geq |U_0| = m$, and there are no edges in G between U_{j-1} and $V(G) \setminus (Z_0 \cup U' \cup U_j)$, so by **L1** we must have $|V(G) \setminus (Z_0 \cup U' \cup U_j)| < m$. Thus $|U_j| \geq |G| - |Z_0| - 2m \geq |G| - \lambda n/2 - 2m > |G|/2$, and therefore $|U_r| \geq |U_j| > |G|/2$.

Similarly, letting $U_0' = U'$ and iteratively taking $U_i' = U_{i-1}' \cup N_{G-Z_0-U}(U_{i-1}')$ for each $1 \leq i \leq r$, we have that $|U_r'| > |G|/2$. Thus, $U_r \cap U_r' \neq \emptyset$, and therefore there is a path from U to U' with length at least 2 and at most ℓ in $G - Z_0$.

Thus, by taking Z_0 to be the internal vertices of a set of maximal internally-vertex-disjoint paths from U to U' in G with length at least 2 and at most ℓ, we see that G contains at least $\lambda n/3\ell$ internally-vertex-disjoint paths from U to U' with length at most ℓ and at least 2. Let $2 \leq j \leq \ell$ be maximal subject to the condition that there are at least $r_j := \lambda n/3^j \ell$ internally-vertex-disjoint paths from U to U' with length j. Such a j exists as $\sum_{j=2}^{\ell} r_j < \lambda n/3\ell$. If $j < \ell$, then let $P_1, \ldots, P_{r_j}$ be such a set of paths, and, for each $i \in [j]$, let x_i be the vertex of the path neighbouring the end-vertex in U (which is not in U' as $j \geq 2$). Note that if $G[\{x_i : i \in [r_j]\}]$ contains a matching with r_{j+1} edges, then we can use this matching along with $P_1, \ldots, P_{r_j}$ to get r_{j+1} internally-vertex-disjoint paths from U to U' in G with length $j+1$, a contradiction. On the other hand, if $G[\{x_i : i \in [r_j]\}]$ contains no matching with r_{j+1} edges, then, removing a maximal matching shows that $G^c[\{x_i : i \in [r_j]\}]$, and hence G^c, contains a copy of $K_{r_j - 2r_{j+1}}$, and hence a copy of $K_{\delta n}^k$ as $\delta \ll 1/\ell, \lambda$, contradiction. Thus, we must have $j = \ell$.

Therefore, altogether, we have shown that, for any disjoint sets $U, U' \subset V(G)$ of size m, there are at least $\lambda n/3^\ell \ell$ internally-vertex-disjoint paths from U to U' with length ℓ. For any such path, the probability all its internal vertices lie in Z is $(1/5)^{\ell-1}$. Thus, as $\delta \ll \lambda, \ell$, by Lemma 2.20, the probability that there are fewer than δn internally vertex-disjoint paths with length ℓ through Z connecting U and U' in G is at most $2\exp(-\lambda n/15^{\ell+1}\ell)$. Therefore, the probability there are disjoint sets $U, U' \subset V(G)$ of size m, with fewer than δn internally-vertex-disjoint paths from U to U' with length ℓ and internal vertices in Z is at most

$$\binom{kn}{m}^2 \cdot 2\exp\left(-\frac{\lambda n}{15^{\ell+1}\ell}\right) \leq \left(\frac{ekn}{m}\right)^{2m} \cdot 2\exp\left(-\frac{\lambda n}{15^{\ell+1}\ell}\right) \leq \left(\frac{ek}{\delta}\right)^{2\delta n} \cdot 2\exp\left(-\frac{\lambda n}{15^{\ell+1}\ell}\right) < 1/3,$$

as required, where the last inequality follows as $1/n \ll \delta \ll \ell, 1/k$. □

Next, we give a simple result to show that, in the sketch above, the set V_0 will have the expansion property in G that we want.

Proposition 6.6. *Let $k \geq 3$ and $1/n \ll \lambda \ll 1/k$. Let $m \leq \lambda n$ and let G be a graph with $(k-1)(n-1)+m$ vertices in which* **L2** *holds. Let $V_0 \subset V(G)$ be a set formed by including each element independently at random with probability $1/5$. Then, with probability more than $2/3$, for any set $U \subset V(G)$ with $|U| = m$, $|N_G(U, V_0)| \geq n/10$.*

Proof. For each set $U \subset V(G)$ with $|U| = m$, we have, by **L2**, that $\mathbb{E}|N_G(U, V_0)| \geq n/6$. Thus, by Lemma 2.20, the probability that there is some set $U \subset V(G)$ with $|U| = m$ and $|N_G(U, V_0)| < n/10$ is at most

$$\binom{kn}{m} \cdot 2\exp\left(-\frac{n}{10^3}\right) \leq \left(\frac{ekn}{m}\right)^m \cdot 2\exp\left(-\frac{n}{10^3}\right) \leq \left(\frac{ek}{\lambda}\right)^{\lambda n} \cdot 2\exp\left(-\frac{n}{10^3}\right) < \frac{1}{3},$$

as required, where we have used that $1/n \ll \lambda \ll 1/k$. $\square$

Using Propositions 6.5 and 6.6, we can now prove Lemma 6.3 using the sketch above, as depicted in Figure 2.

Proof of Lemma 6.3. Let L, δ and ℓ satisfy $\varepsilon \ll 1/L \ll \delta \ll 1/\ell \ll \lambda$, and let $d = 4\Delta$. Let $V(G) = Z \cup V_0 \cup V_1$ be a partition chosen by selecting the set for each $v \in V(G)$ independently at random so that $\mathbb{P}(v \in Z) = \mathbb{P}(v \in V_0) = 1/5$ and $\mathbb{P}(v \in V_1) = 3/5$.

By a simple application of Lemma 2.20, as $|G| \geq 2n-1$ we have that, with probability greater than $2/3$, $|V_1| \geq 11n/10$. Therefore, by **L1** and **L2**, using Propositions 6.5 and 6.6, we can take a choice of partition $V(G) = W \cup V_0 \cup V_1$ for which the following hold.

M1 $|V_1| \geq 11n/10$.

M2 For every pair of vertex disjoint sets $U_1, U_2 \subset V(G)$ with size m, there are at least δn internally vertex disjoint paths from U_1 to U_2 with length ℓ through Z.

M3 For each $U \subset V(G)$ with $|U| = m$, $|N(U, V_0)| \geq n/10$.

Now, let $X_0 \subset V_0$ have size $n/20$ (possible by **M3**), and note that, by **M3**, for each $U \subset V(G)$ with $|U| = m$, $|N(U, V_0 \setminus X_0)| \geq n/20$. Then, similarly to the proof of Proposition 2.13, there is a set $W \subset V(G)$ with $|W| < m$ such that, for each $U \subset V(G) \setminus W$ with $|U| \leq m$, $|N_G(U, V_0 \setminus (X_0 \cup W))| \geq 3d|U|$. Let $V_0' = V_0 \setminus W$ and $X = X_0 \setminus W$. Then, by Proposition 2.15, we have that $I(X)$ is (d, m)-extendable in $G[V_0']$. Let $n_0 = |V_0'| - n/10$, and note that $G[V_0']$ is (m, n_0)-joined by **M3** and

$$|V_0'| = n_0 + n/10 \geq n_0 + |X| + (2d+2)m + n/40. \tag{6.4}$$

Now, take T and arbitrarily select a leaf t_1 of T. For each $i = 2, \ldots, n$ in turn, if possible let t_i be a neighbour of t_{i-1} in T which is not in $\{t_1, \ldots, t_{i-2}\}$, and otherwise, let t_i be any vertex of T not in $\{t_1, \ldots, t_{i-2}\}$ with a neighbour in $\{t_1, \ldots, t_{i-2}\}$. For each $2 \leq i \leq n$, let s_i be the neighbour of t_i in $T[\{t_1, \ldots, t_{i-1}\}]$. We will build the embedding ϕ of T by embedding $t_1, t_2, \ldots, t_n$ in this order. To initialise, set $I_a = \{1\}, I_b = \emptyset$ and $\phi(t_1) = v_1$ for an arbitrary vertex $v_1 \in X$. Then, beginning with $j = 2$, carry out the process whose step j is as follows.

a) If $j \leq n - L$ and $t_j, \ldots, t_{j+L}$ all have degree 2 in T, and there is some $v_{j+L} \in X \setminus \phi(\{t_1, \ldots, t_{j-1}\})$ for which there is a $\phi(s_j), v_{j+L}$-path $\phi(s_j)v_j \ldots v_{j+L}$ in G with internal vertices not in $V_0 \cup \phi(\{t_1, \ldots, t_{j-1}\})$ and at most 2ℓ vertices in Z, then let $\phi(t_i) = v_i$ for each $j \leq i \leq j + L$, and add $j + L$ to I_a.

If $j + L = n$ then stop, otherwise proceed to step $j + L + 1$.

b) Otherwise, if s_j satisfies $\phi(s_j) \in V_0'$, and there is some $v_j \in V_0' \setminus (X \cup \{t_1, \ldots, t_{j-1}\})$ such that $I(X) \cup \phi(T[\{t_i : i \in I_a \cup I_b\}]) + \phi(s_j)v_j$ is (d, m)-extendable in $G[V_0']$, then let $\phi(t_j) = v_j$ and add j to I_b.

If $j = n$, then stop, otherwise proceed to step $j + 1$.

c) If neither of these cases occur, then stop.

Assume we are at the start of step j of this process, note the following. If $j \leq n - L$ and $t_j, \dots, t_{j+L}$ all have degree 2 in T, then by the ordering of the vertices in T, we have that $T[\{t_j, \dots, t_{j+L}\}]$ is a path with length L. Also, I_a is the set of indices $i \in [j-1]$ such that $\phi(t_i) \in X$, I_b is the set of indices $i \in [j-1]$ such that $\phi(t_i) \in V_0' \setminus X$, and, if $i \in [j-1]$ and t_i is adjacent to some $t_{j'} \in V(T)$ with $j' \geq j$, then $\phi(t_i) \in V_0'$ and $i \in I_a \cup I_b$. We now further analyse this process via the following three claims and show that it will only stop at a step j if $j + L = n$ or $j = n$, and that, in either case, this implies that we have embedded a copy of T in G.

Claim 6.7. *If the process stops at step j with $j \neq n$ and $j + L \neq n$, then $|I_b| \geq n/50$.*

Proof of Claim 6.7. Suppose for a contradiction, that $|I_b| < n/50$. From the observations above, we have $\phi(s_j) \in V_0'$. Furthermore, using $|I(X) \cup \phi(T[\{t_i : i \in I_a \cup I_b\}])| = |X| + |I_b|$ and (6.4), we have

$$|V_0'| \geq |I(X) \cup \phi(T[\{t_i : i \in I_a \cup I_b\}])| + (2d+2)m + n_0 + 1.$$

As $I(X) \cup \phi(T[\{t_i : i \in I_a \cup I_b\}])$ is (d,m)-extendable in $G[V_0']$, which is (m, n_0)-joined, and the degree of $\phi(s_i)$ is at most $\Delta - 1$ in $I(X) \cup \phi(T[\{t_i : i \in I_a \cup I_b\}])$, by Lemma 2.16, there is $v_j \in V_0' \setminus (X \cup \{t_1, \dots, t_{j-1}\})$ adjacent to $\phi(s_i)$, such that $I(X) \cup \phi(T[\{t_i : i \in I_a \cup I_b\}]) + \phi(s_j)v_j$ is (d,m)-extendable in $G[V_0']$. This contradicts that the process stopped at step j as we could have carried out **b)**. ⊡

Claim 6.8. $|X \cap \phi(\{t_1, \dots, t_{j-1}\})| \leq n/100$ *and* $|Z \cap \phi(\{t_1, \dots, t_{j-1}\})| < \delta n/4$.

Proof of Claim 6.8. Note that, for each $j \geq 2$, $\phi(t_j) \in X$ only if step $j - L$ is carried out via part **a)**, and exactly 1 vertex is embedded into X in this step. Thus, $|X \cap \phi(\{t_1, \dots, t_{j-1}\})| \leq 1 + (n-1)/L \leq n/100$. Similarly, the only vertices embedded into Z are embedded via part **a)**, where at most 2ℓ vertices are used each step. Thus, $|Z \cap \phi(\{t_1, \dots, t_{j-1}\})| \leq 2\ell(n-1)/L < \delta n/4$. ⊡

Claim 6.9. *If $|I_b| \geq n/50$, then there are vertex disjoint sets $U_1, U_2 \subset V_0'$ with size m such that there is no path in G from U_1 to U_2 of length L with internal vertices not in $V_0' \cup \phi(\{t_1, \dots, t_{j-1}\})$ and at most 2ℓ vertices in Z.*

Proof of Claim 6.9. Suppose $|I_b| \geq n/50$. Let $J \subset I_b$ be the set of $j \in I_b$ such that $t_j, \dots, t_{j+L}$ all have degree 2 in T. Suppose for a contradiction that $|J| < |I_b|/2$. For each $j \in J \setminus I_b$, there is some $j \leq i \leq j + L$ such that t_i either has degree 1 or degree at least 3 in T, so in total there are at least $|I_b|/2(L+1) \geq n/100(L+1)$ such indices i. As T has at most $10\Delta^2\varepsilon n$ leaves and $\varepsilon \ll 1/L$, T must then have fewer leaves than vertices with degree at least 3. However,

$$2(n-1) = 2e(T) = \sum_{i \in [n]} d_T(t_i) \geq |\{i : d_T(t_i) = 1\}| + 2(n - |\{i : d_T(t_i) = 1\}|) + |\{i : d_T(t_i) \geq 3\}|,$$

so it follows that T has at least as many leaves as vertices with degree at least 3, contradiction. Thus, $|J| \geq |I_b|/2 \geq n/100 \geq \Delta m$.

Let $U_1 \subset \phi(\{s_i : i \in J\})$ have size m, which is possible as $|J| \geq \Delta m$ and let $U_2 \subset X \setminus \phi(\{t_1, \dots, t_{j-1}\})$ have size m (possible by Claim 6.8). Suppose there exists a path in G with length L connecting some $\phi(s_i) \in U_1$ with some vertex in U_2, such that all its internal vertices are not in $V_0' \cup \phi(\{t_1, \dots, t_{j-1}\})$ and at most 2ℓ of them are in Z, then we could have carried out step i with **a)** to embed t_i. But as $i \in J \subset I_b$, we actually carried out step i with **b)** instead, a contradiction. Hence, such a path does not exist, proving the claim. ⊡

Now, suppose for a contradiction that $|I_b| \geq n/50$. Using Claim 6.9, take vertex-disjoint sets $U_1, U_2 \subset V_0'$ with size m such that there is no path in G from U_1 to U_2 of length L with internal vertices not in $V_0' \cup \phi(\{t_1, \dots, t_{j-1}\})$ and at most 2ℓ of them in Z. Let $V_1'' = V_1 \setminus \phi(\{t_1, \dots, t_{j-1}\})$, so that $|V_1''| \geq n/10$ by **M1**. By Theorem 6.4, as G^c, and hence $G^c[V_1'']$, contains no copy of $K_{\varepsilon n}^{k-1} \times K_m^c$, there must be a path with length $2Lm$ in $G[V_1'']$. On this path, we can find vertex-disjoint paths $Q_1, \dots, Q_{2m}$ in $G[V_1'']$, each with length $L - 2\ell$. For each $i \in [2m]$, let x_i and y_i be the end vertices of Q_i.

Let $I_1 \subset [2m]$ be the set of $i \in [2m]$ for which there are at least ℓ vertex-disjoint paths of length ℓ from x_i to U_1 with internal vertices in $Z \setminus \phi(\{t_1, \dots, t_{j-1}\})$. Then, there are at most $\ell \cdot |[2m] \setminus I_1| < \delta n/2$ vertex-disjoint paths of length ℓ from $\{x_i : i \in [2m] \setminus I_1\}$ to U_1 with internal vertices in $Z \setminus \phi(\{t_1, \dots, t_{j-1}\})$. By Claim 6.8 and **M2**, we must then have that $|[2m] \setminus I_1| < m$, so that $|I_1| \geq m+1$. Similarly, if $I_2 \subset [2m]$ is the set of $i \in [2m]$ for which there are at least ℓ vertex-disjoint paths of length ℓ from y_i to U_2 with internal vertices in $Z \setminus \phi(\{t_1, \dots, t_{j-1}\})$, then $|I_2| \geq m+1$. Hence, there exists some $i \in I_1 \cap I_2$. Let P_1 be a path of length ℓ from x_i to U_1 with internal vertices in $Z \setminus \phi(\{t_1, \dots, t_{j-1}\})$. Since there are at least ℓ vertex disjoint paths of length ℓ from y_i to U_2 with internal vertices in $Z \setminus \phi(\{t_1, \dots, t_{j-1}\})$, we can find one, say P_2, that is disjoint from P_1. Then P_1, Q_i and P_2 attached together form a path in G from U_1 to U_2 of length L with internal vertices not in $V_0' \cup \phi(\{t_1, \dots, t_{j-1}\})$ and at most 2ℓ of them in Z, a contradiction.

Thus, we must have that $|I_b| < n/50$. Therefore, by Claim 6.7, the process stops with $j = n$ or $j = n - L$, and in either case the process has then embedded a copy of T. Thus, G contains a copy of T, as required. $\square$

6.4 Proof of Theorem 2.1

Finally, using Lemmas 6.1, 6.2 and 6.3, we can prove Theorem 2.1 from Theorem 2.2 by induction.

Proof of Theorem 2.1. For each Δ, the proof is by induction on $k \geq 2$. When $k = 2$, we have shown that the required $\mu_{\Delta,2}$ exists by Theorem 2.2, so let us assume that $k \geq 3$ and that the required $\mu_{\Delta,k'}$ exist for each $2 \leq k' \leq k-1$. Let $\mu \leq \min\{\mu_{\Delta,k'} : 2 \leq k' \leq k-1\}$, $\mu \ll 1/\Delta, 1/k$, and let ε and λ satisfy $\varepsilon \ll \lambda \ll \mu$. Note that we can assume that $1/n \ll \varepsilon$, for if we have proved it for all $n \geq n_0$ with $1/n_0 \ll \varepsilon$, then we can reduce ε to $1/n_0$ to get a result for all n. Note that this choice of μ implies the following property.

N1 For any $2 \leq k' \leq k-1$ and $m' \leq \mu n$, $R(T, K_{\mu n}^{k'-1} \times K_{m'}^c) = (k'-1)(n-1) + m'$.

We will show that we can take $\mu_{\Delta,k} = \varepsilon$.

For this, let $m \leq \varepsilon n$ and let T be any n-vertex tree with $\Delta(T) \leq \Delta$. Let G be a graph on $(k-1)(n-1)+m$ vertices such that G contains no copy of T and G^c contains no copy of $K_{\varepsilon n}^{k-1} \times K_m^c$. Similarly to the start of the proof of Lemma 6.2 (with **K3**), we can assume we have the following property.

N2 For every $U \subset V(G)$ with $|U| = m$, we have $|N_G(U)| \geq (1-\lambda)n$.

Furthermore, if $U \subset V(G)$, $|U| = \varepsilon n$ and $|U \cup N_G(U)| < n$, then we have $|G - U - N_G(U)| \geq (k-2)(n-1)+m$, and, thus, as $\varepsilon \ll \mu$, the complement of $G - U - N_G(U)$ contains a copy of $K_{\varepsilon n}^{k-2} \times K_m^c$, which, with all the edges from this to U, forms a copy of $K_{\varepsilon n}^{k-1} \times K_m^c$ in G^c, a contradiction. Thus, we can assume that,

N3 For every $U \subset V(G)$ with $|U| = \varepsilon n$, we have $|U \cup N_G(U)| \geq n$.

Now, if T has more than $10\Delta^2 \varepsilon n$ leaves, then, by **N3** and Lemma 6.1, G contains a copy of T, a contradiction. Thus, we can assume that T has fewer than $10\Delta^2 \varepsilon n$ leaves. Therefore, by **N1** and Lemma 6.2, there must be no partition $V(G) = V_0 \cup V_1 \cup V_2$ such that $e_G(V_1, V_2) = 0$, $|V_1|, |V_2| \geq m$ and $|V_0| \leq \lambda n$. Finally, then, by **N2** and Lemma 6.3, G contains a copy of T or G^c contains a copy of $K_{\varepsilon n}^{k-1} \times K_m^c$, a contradiction.

Therefore, for every $m \leq \varepsilon n$, every n-vertex tree T with $\Delta(T) \leq \Delta$, and every graph G with $(k-1)(n-1) + m$ vertices, G contains a copy of T or G^c contains a copy of $K_{\varepsilon n}^{k-1} \times K_m^c$, so that setting $\mu_{\Delta,k} = \varepsilon$ completes the proof. $\square$

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