Global rigidity of random graphs in $\mathbb{R}$

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Abstract

We investigate the problem of reconstructing a set $P \subseteq \mathbb{R}$, where the only information available about P consists of the pairwise distances between some of the pairs of points. More precisely, we examine which properties of the graph G of known distances, defined on the vertex set P, ensure that P can be uniquely reconstructed up to isometry. Deciding whether an input graph reconstructs every point set has previously been proven to be computationally difficult, and thus any sufficient and necessary property is likely to be involved. Motivated by this, we derive a simple sufficient property which applies to typical random graphs. As a corollary, we prove that as soon as the random graph process has minimum degree 2, with high probability it can reconstruct the distances among any point set in $\mathbb{R}$, which resolves a conjecture of Benjamini and Tzalik. We also study the feasibility and limitations of reconstructing the distances among almost all points using much sparser random graphs. In doing so, we resolve a question posed by Girão, Illingworth, Michel, Powierski, and Scott.

1 Introduction

We consider a set $P \subseteq \mathbb{R}^d$ of (distinct) points and we are interested in the distances between any two of them, but we only have access to the distances between some of them. Which properties of the graph G of known distances on the vertex set P are sufficient for the "reconstruction" of P, up to isometry? One can imagine this problem arising naturally in various scenarios when we are interested in monitoring the relative positions of a set of agents (say ships in the ocean or birds in the sky) moving in space [13]. These agents are equipped with sensors that allow certain pairs to measure the distance between them, though measuring all pairwise distances may be impractical. This leads to several key questions: When can we recover the geometric positions of all agents relative to each other? How should we choose G so that we are able to recover the relative positions no matter how the agents are placed? What if we are content with recovering the relative positions of only almost all of the agents?

The complete graph on any such P of course fully reconstructs P, but the aim here would rather be to reconstruct from less than complete information. To be able to do that in $\mathbb{R}^d$ when $d \geq 2$, one needs to impose some restriction on P, as otherwise there are examples which show that if G is missing just one (carefully chosen) edge, a full reconstruction of P is not possible. For example, consider the configuration with $n-2$ points on a line and two points outside of the line. Then we cannot decide whether these two points lie on the same side of the line or not, unless we are given the distance between them. It turns out that if one restricts to so-called *generic* point sets P, the coordinates of which are algebraically independent over the rationals, then whether or not P is reconstructible from G depends only on the combinatorial properties of G and not on P. This case has been extensively studied and has a rich mathematical theory (e.g. see [2, 10, 11, 12, 14, 16]; for a thorough introduction to the topic, see [15]).

When $d = 1$ however, unlike in the case of higher dimensions, there are no clear obstacles which justify restricting to generic, as opposed to arbitrary, point sets. It is a folklore result (e.g. see [15,

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Chapter 63]) that the known-distance graph G reconstructs any generic point set $P \subseteq \mathbb{R}$ if and only if it is 2-connected. There is a rich family of two-connected graphs however which do not reconstruct certain non-generic point sets in $\mathbb{R}$ (the four-cycle to begin with). That is, there are many graphs which can reconstruct any generic point set in $\mathbb{R}$ but not any non-generic point set in $\mathbb{R}$. Garamvölgyi [7] in fact proved that this family is pretty complex, as it is in co-NP to decide whether a given graph reconstructs *any* set of distinct points in $\mathbb{R}$, while in higher dimensions the family of graphs which reconstruct every generic point set but not every non-generic point set includes, for example, graphs of arbitrarily large connectivity [8].

This difficulty makes the study of the typical behaviour in this context particularly interesting. Recently Benjamini and Tzalik [3] initiated the study of reconstructability of the Erdős-Rényi random graph $G(n, p)$. For any point set $P \subseteq \mathbb{R}$ with $|P| = n$, they showed that for some constant C it holds that if the graph G of known distances is distributed as $G(n, C \log n/n)$, then with high probability (w.h.p.) P is reconstructible from G. This result is of course best possible up to the constant factor C as it is well-known that for $p < (\log n + \log \log n)/n$ the random graph $G(n, p)$ w.h.p. is not 2-connected, hence does not even reconstruct generic point sets. Another important aspect of the above result is that we are *first* given P in $\mathbb{R}$ and *only after that* we randomly generate the graph of known distances, which, with high probability, is proven to reconstruct all distances within that specific set P.

In both of these directions Benjamini and Tzalik [3] conjectured a significant strengthening of their result: namely, that (i) a hitting time result should also hold, i.e. that the reconstruction property for a particular P should likely be true from the very moment the random graph process becomes 2-connected, and (ii) at that point, not only does the random graph reconstructs the given point set, but it reconstructs any such set. Recently Girão, Illingworth, Michel, Powierski, and Scott [8] proved (i). In our first main result, stated below, we prove (ii) using a different and simpler approach. We also study the possibilities/limitations of the reconstruction of a linear sized portion of the point set.

1.1 New results.

Terminology. Consider a random graph process $\{G_m\}_{m\geq 0}$ on the vertex set $[n]$, for some $n \in \mathbb{N}$, where G_0 is an empty graph and each G_i is formed from G_{i-1} by adding a new edge uniformly at random. Let $\tau := \tau_2$ denote the smallest m such that $\delta(G_m) \geq 2$.
A *(bar-and-joint) framework (in $\mathbb{R}^d$)* is a pair (G, f) where G is a simple graph and $f : V(G) \to \mathbb{R}^d$ maps the vertices into the d-dimensional euclidean space. For a graph $G = (V, E)$ and an injective function $f : V \to \mathbb{R}^d$, we call the framework (G, f) *globally rigid* if for every injective function $g : V \to \mathbb{R}^d$ with the property that $||f(x) - f(y)|| = ||g(x) - g(y)||$ for every edge $xy \in E$, we also have $||f(x) - f(y)|| = ||g(x) - g(y)||$ for every pair $x, y \in V$. To express this in our informal description above, we said that "G reconstructs the point set $f(V)$ (up to isometry)". We call a graph G *globally rigid (in $\mathbb{R}^d$)* if for every injective function $f : V \to \mathbb{R}^d$ the framework (G, f) is globally rigid.

Remark. We note that the term "globally rigid" in the literature is often used in the sense that *generic* embeddings are reconstructible. In contrast, as introduced above, "globally rigid" in our paper will always mean that *injective* embeddings are reconstructible. Garamvölgyi [7] refers to this as "injectively globally rigid". As this is the only type of rigidity we deal with, we omit the adjective.

Using this terminology, the theorem of Girão, Illingworth, Michel, Powierski, and Scott [8], mentioned earlier, can be restated as follows.

Theorem 1.1 ([8])**.** *For every injective function $f: [n] \to \mathbb{R}$ the framework (G_τ, f) is globally rigid w.h.p.*

The proof of this result follows the approach laid down by [3]. In our first main result we prove that G_τ with high probability not only reconstructs the given embedding f but reconstructs any embedding. This confirms the conjecture of Benjamini and Tzalik. Our proof strategy is different (and simpler) from the previous ones.

Theorem 1.2. *G_τ is globally rigid in $\mathbb{R}$ w.h.p.*

As the hitting time of 2-connectivity is equal to τ w.h.p., our result shows that on the level of "typical" graphs there is no difference in $\mathbb{R}$ between generic global rigidity and our less restrictive injective version. For the same reason, for random graphs sparser than G_τ one cannot hope in general for the reconstruction of the full point set. Were we content ourselves with reconstructing only almost every point, then we can do much better. The bottleneck here again turns out to be 2-connectivity. It is well-known that a 2-connected component of size $(1 - o(1))n$ emerges in $G(n, p)$ when $p = \omega(1/n)$. Girão et al. [8] showed that in the same regime it also holds that given any injective $f\colon [n] \to \mathbb{R}$, the random graph $G(n,p)$ w.h.p. reconstructs f on some subset $V' \subseteq [n]$ of size $|V'| = (1-o(1))n$.

They asked whether $1/n$ is also the threshold for the property that $G(n,p)$ reconstructs a constant fraction of points for *any* injective function f. Using our novel approach developed for Theorem 1.2 we can show that this holds in a much stronger form. Namely, one can always reconstruct the same subset of vertices, independent of f, the size of which is a fraction of n arbitrarily close to 1.

Theorem 1.3. *For every $\varepsilon > 0$ there exists $C \in \mathbb{R}$ such that in the random graph $G \sim G(n, C/n)$ w.h.p. there exists a subset $V' \subseteq [n]$ of size $|V'| \geq (1-\varepsilon)n$ for which the induced subgraph $G[V']$ is globally rigid in $\mathbb{R}$.*

Moreover, Girão et al. [8] conjectured that for a given injective embedding it holds w.h.p. right after the point $p = 1/n$ of the phase transition for the emergence of the giant 2-connected component in $G(n,p)$, that a positive fraction of the points can also be reconstructed.

Conjecture 1.4. *Let $f : [n] \to \mathbb{R}$ be an arbitrary injective function and $\varepsilon > 0$. Then for every $p \geq (1+\varepsilon)/n$, in the random graph $G \sim G(n,p)$ w.h.p. there exists a subset $V' \subseteq [n]$ of size $|V'| = \Omega_\varepsilon(n)$ such that the restricted framework $(G[V'], f|_{V'})$ is globally rigid in $\mathbb{R}$.*

We do not settle this conjecture, but we show that its strengthening to global rigidity does not hold.

Theorem 1.5. *There exists $\gamma > 0$ such that for any $p < 1.1/n$, the random graph $G \sim G(n,p)$ w.h.p. satisfies that for every subset $V' \subseteq [n]$ of size $|V'| \geq \gamma \log n$, the induced subgraph $G[V']$ is not globally rigid in $\mathbb{R}$.*

We prove Theorem 1.2 and Theorem 1.3 in Section 2. In Section 3 we prove Theorem 1.5, before finishing with some open problems in Section 4.

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2 Criteria for global rigidity

The following lemma is the crux of our proofs of Theorem 1.2 and Theorem 1.3, which are then derived as simple corollaries from it.

Lemma 2.1. *Let G be a graph with $V(G) = [n]$, and suppose it satisfies the following two properties:*

(P1) For every disjoint $U, W \subseteq V(G)$ of size $|U|, |W| \geq n/15$ there is an edge in G between U and W.

(P2) For every $U \subseteq V(G)$ of size $n/15 \leq |U| < n$, there exists a vertex $v \in V(G) \setminus U$ with at least two neighbors in U.

Then G is globally rigid in $\mathbb{R}$.

Proof. The result holds trivially for $n = 1$, thus we assume $n > 1$. Let f and $g : [n] \to \mathbb{R}$ be two injective functions such that $|f(x) - f(y)| = |g(x) - g(y)|$ for every edge $xy \in E(G)$. We show that the same equality holds for any two vertices $x, y \in V(G)$, i.e. f and g are isometric. Equivalently, this means that there exists $a \in \{1, -1\}$ and $b \in \mathbb{R}$ such that $f = af' + b$. Let

$$L_f := \{i \in [n] : |\{x \in [n] : f(i) < f(x)\}| \geq \lceil n/2 \rceil\}$$
$$R_f := \{i \in [n] : |\{x \in [n] : f(x) < f(i)\}| \geq \lceil n/2 \rceil\}$$

be the *left-half* and *right-half* of f (omitting the middle vertex when n is odd), and define L_g and R_g analogously.

We can assume without loss of generality that $|L_f \cap L_g| \geq \lceil (|L_f| - 1)/2 \rceil \geq \lceil (n-3)/4 \rceil$. Otherwise we consider instead the function $-g$, which is isometric to g, and use that $L_{-g} = R_g$. As $n > 1$, by

(P2) we necessarily have $n > 15$ (otherwise G does not satisfy (P2)), thus $|L_f \cap L_g| > n/5$. Then $|R_f \cap R_g| = |R_f| + |R_g| - |R_f \cup R_g| \geq n - 1 - |\overline{(L_f \cap L_g)}| \geq n/5$. Set $L := L_f \cap L_g$ and $R := R_f \cap R_g$.

We first prove that the induced bipartite graph $G[L, R]$ contains a connected component spanning at least $n/15$ vertices. Let $C^{(1)}, \ldots, C^{(k)}$ be any ordering of the connected components of $G[L, R]$. Toward a contradiction, assume that each connected component contains fewer than $n/15$ vertices, i.e. for all $j \in [k]$, we have $|C^{(j)}| < n/15$. Let $i > 1$ be the smallest index such that

$$\sum_{j=1}^{i} |C^{(j)} \cap L| \geq n/15 \quad \text{or} \quad \sum_{j=1}^{i} |C^{(j)} \cap R| \geq n/15.$$

Without loss of generality assume that $\sum_{j=1}^{i} |C^{(j)} \cap R| \leq \sum_{j=1}^{i} |C^{(j)} \cap L|$. Then, using $|C^{(j)}| < n/15$ for all $j \in [k]$, by the minimality of i we have

$$\sum_{j=1}^{i} |C^{(j)} \cap R| \leq \sum_{j=1}^{i} |C^{(j)} \cap L| < 2n/15.$$

As $\sum_{i=1}^{k} |C^{(j)} \cap R| = |R| \geq n/5$, we then have

$$\sum_{j=i+1}^{k} |C^{(j)} \cap R| \geq n/15.$$

Then by (P1) there exists an edge between $\cup_{j=1}^{i} C^{(j)} \cap L$ and $\cup_{j=i+1}^{k} C^{(j)} \cap R$, which contradicts the assumption that $C^{(1)}, \ldots, C^{(k)}$ are the connected components of $G[L, R]$.

Let C be the vertices of the largest connected component of $G[L, R]$. As we have just showed, $|C| \geq n/15$. Let $y_1 \in C \cap L$ be an arbitrary vertex and let $g' = g - g(y_1) + f(y_1)$, i.e. the translation of g that agrees with f on y_1. As this is not changing the relative order with respect to g, we have $L_g = L_{g'}$ and $R_g = R_{g'}$. Let $U := \{u \in [n] : f(u) = g'(u)\}$ be the set of vertices on which f agrees with g'. By the definition, we have $y_1 \in U$. We claim that the whole connected component C is contained in U. This is because for any vertex $x \in U$ and edge $xy \in E(G)$ of C, we also have $y \in U$: Suppose first that $x \in L_f \cap L_{g'}$ and $y \in R_f \cap R_{g'}$; the other case is analogous. Since x is in the left-half of f, y is in the right-half of f, we have $f(y) = f(x) + |f(x) - f(y)|$. Similarly, we obtain $g'(y) = g'(x) + |g'(x) - g'(y)|$. Using that $|f(x) - f(y)| = |g'(x) - g'(y)|$ and $x \in U$, we conclude that $f(y) = g'(y)$, so $y \in U$.

Now $C \subseteq U$ implies $|U| \geq n/15$. If $U = [n]$ we are done. Otherwise (P2) can be applied and we take a vertex $v \in V(G) \setminus U$ which has two neighbors u_1, u_2 in U. Assume, by relabelling if necessary, that $f(u_1) = g'(u_1) < f(u_2) = g'(u_2)$. The f-value of v is determined by $f(u_1), f(u_2), |f(u_1) - f(v)|$ and $|f(u_2) - f(v)|$. Indeed, depending on whether $f(u_2)$, $f(u_1)$ or $f(v)$ is in between the other two, $f(v)$ is equal to $f(u_1) + |f(u_1) - f(v)| = f(u_2) + |f(u_2) - f(v)|$, $f(u_1) - |f(u_1) - f(v)| = f(u_2) - |f(u_2) - f(v)|$, or $f(u_1) + |f(u_1) - f(v)| = f(u_2) - |f(u_1) - f(v)|$, respectively. The g'-value of v is determined by $g'(u_1), g'(u_2), |g'(u_1) - g'(v)|$ and $|g'(u_2) - g'(v)|$ analogously. Finally, since $f(u_1) = g'(u_1)$ and $f(u_2) = g'(u_2)$, the values $f(v)$ and $g'(v)$ coming from the analogous formulas also agree. That means $v \in U$, contradicting $v \in V(G) \setminus U$. Thus, $U = V(G)$, and therefore, as $f = g'$, f and g are isometric. □

2.1 Applications

We need the following simple property of random graphs.

Lemma 2.2. *For every $\varepsilon > 0$ there exists $C > 0$ such that if $m \geq Cn$, then $G \sim G(n, m)$ w.h.p. has the following property:*

(P3) For every disjoint $X, Y \subseteq V(G)$ of size $|X|, |Y| \geq \varepsilon n$, there exists an edge between X and Y in G.

Proof. For fixed X and Y, the probability that there is no edge between X and Y is

$$\binom{\binom{n}{2} - |X||Y|}{m} \Big/ \binom{\binom{n}{2}}{m} \leq e^{-|X||Y|m/n^2} \leq e^{-\varepsilon^2 Cn}.$$

There are at most 2^{2n} ways to choose X and Y, thus, for $C > 2/\varepsilon^2$, w.h.p. this bad event does not happen for any such pair of sets. □

With Lemma 2.1 and Lemma 2.2 at hand, the proofs of Theorems 1.2 and 1.3 are straightforward.

Proof of Theorem 1.2. For the proof of Theorem 1.2 we check that w.h.p. both (P1) and (P2) hold for G_τ, so that the result follows directly by Lemma 2.1.

Let C be a constant given by Lemma 2.2 for $\varepsilon = 1/31$. It is well known [4] that w.h.p. $\tau \geq Cn := m$ (with C as given by Lemma 2.2, or, indeed, any constant C). As G_m is uniformly distributed among all graphs with n vertices and exactly m edges, by Lemma 2.2 we have that w.h.p. (P3) holds in G_m. Since (P3) is monotone, it also holds in G_τ.

Property (P3) is straightforwardly stronger than (P1) and also implies (P2) in the case $n/15 \leq |U| \leq n/2$. Indeed, for the latter let $S \subseteq V(G) \setminus U$ be a subset of size εn. By (P3) we have

$$|N(S) \cap U| \geq |U| - \varepsilon n > |S|,$$

thus there exists a vertex in S with two neighbors in U. The remaining case $|U| > n/2$ of the property (P2) is proven to hold w.h.p., for example, in [12, Proposition 2.3]. □

Proof of Theorem 1.3. For convenience we prove our result for the $G(n, m)$ random graph model where a graph is chosen uniformly at random among all labeled graphs with n vertices and m edges. Namely, for every $\varepsilon > 0$ we show that there exists $C \in \mathbb{R}$ such that for $G \sim G(n, Cn)$ w.h.p. both (P1) and (P2) hold. The equivalent statement for $G(n, p)$ follows by [6, Theorem 1.4].

We can assume $\varepsilon > 0$ is sufficiently small, take C from Lemma 2.2 applied to ε and let $m \geq Cn$. By Lemma 2.2, $G \sim G(n, m)$ w.h.p. has the property (P3). This immediately implies (P1), thus to apply Lemma 2.1 we just need to find a large subset $V' \subseteq V(G)$ such that $G' = G[V']$ satisfies (P2). We define $V' := V(G) \setminus A$, where $A \subseteq V(G)$ is a largest subset such that $|A| \leq \varepsilon n$ and $|N(A)| \leq |A|$.

To check (P2) for a subset $U \subseteq V'$ of size $|V'|/15 \leq |U| < |V'| - \varepsilon n$ we consider a subset $S \subseteq V' \setminus U$ of size $\varepsilon n \leq |V' \setminus U|$. Applying (P3) we have

$$|N(S) \cap U| \geq |U| - \varepsilon n > |S|,$$

which verifies that there is a vertex in S with two G'-neighbors in U, for otherwise $|N(S) \cap U| \leq |S|$.

If $U \subseteq V'$ is of size $|U| \geq |V'| - \varepsilon n$, then we claim that for $S = V' \setminus U$ we have $|N(S) \setminus A| > |S|$, which in turn implies that some vertex of S has two neighbors in U. Otherwise we have

$$|N(A \cup S)| \leq |A| + |S| = |A \cup S|,$$

which implies $\varepsilon n < |A \cup S| \leq 2\varepsilon n$ by the maximality of A. We can then apply (P3) to obtain

$$|N(A \cup S)| \geq n - |A \cup S| - \varepsilon n > |A \cup S|,$$

a contradiction. □

3 Sparse random graphs are typically far from globally rigid

The next lemma gives a general condition under which a graph is not globally rigid in $\mathbb{R}$.

Lemma 3.1. *Suppose the vertex set of a graph $G \neq K_2$ can be partitioned into non-empty sets A and B such that the maximum degree of $G[A, B]$ is at most 1. Then G is not globally rigid.*

Proof. We define two non-isometric injective functions f and $g : V(G) \to \mathbb{R}$ such that $|f(x) - f(y)| = |g(x) - g(y)|$ for every edge $xy \in E(G)$. For f we embed A injectively in $[0, 1]$ arbitrarily, and, for every $b \in B$ such that there exists $a \in A$ with $ab \in E(G)$, we embed b at $f(a) + 10$. We embed the rest of B injectively and arbitrarily inside $[10, 11] \setminus (f(A) + 10)$. We now define the injective g such that $g|_A = f|_A$ and for every $b \in B$, we set $g(b) = f(b) - 20$. Note that, for an edge $xy \in E(G)$ inside A or inside B the property $|f(x) - f(y)| = |g(x) - g(y)|$ is immediate. For and edge $ab \in E(G)$ with $b \in B$ and $a \in A$, we have that $g(b) = f(a) - 10$, so $|g(a) - g(b)| = 10 = |f(a) - f(b)|$. Furthermore, for every $a \in A$ and $b \in B \setminus N(a)$, we have

$$|g(a) - g(b)| - |f(a) - f(b)| = (f(a) - (f(b) - 20)) - (f(b) - f(a)) = 2(10 - |f(a) - f(b)|) \neq 0,$$

since $f(b) \notin f(A) + 10$. This shows that f and g are not isometric, since $G[A, B]$ is not a complete bipartite graph. □

The next lemma gives us a structural property of sparse random graphs from which we deduce Theorem 1.5. The proof is based on a modified version of the classical study of the sizes of the connected components of sparse random graphs using comparisons to Galton-Watson processes. We first recall the following Chernoff bounds (see, for instance, Theorem 2.1 in [9]) as well as Azuma's inequality (see, for instance, [17], for the result and the definition of a submartingale).

Lemma 3.2. *Let X be a binomial random variable with expected value μ. Then, for any $\delta \in (0,1)$, we have*

(i) $\Pr(X < (1-\delta)\mu) \leq e^{\frac{-\delta^2\mu}{2}}$

(ii) $\Pr(X > (1+\delta)\mu) \leq e^{\frac{-\delta^2\mu}{2+\delta}}$

Lemma 3.3. *Let $(X_i)_{i\geq 0}$ be a submartingale and let $c_i > 0$ for each $i \geq 1$. If $|X_i - X_{i-1}| \leq c_i$ for each $i \geq 1$, then, for each $n \geq 1$,*

$$\Pr(X_n \leq X_0 - t) \leq \exp\left(-\frac{t^2}{2\sum_{i=1}^n c_i^2}\right).$$

Lemma 3.4. *Let $G \sim G(n,p)$ for $p \leq \frac{1.1}{n}$. Then, w.h.p., for each $v \in V(G)$ there is some $A_v \subset V(G)$ with $v \in A_v$ and $|A_v| \leq 6000 \log n$ such that the maximum degree of $G[A_v, V(G) \setminus A_v]$ is at most 1.*

Proof. Note that we can suppose for the rest of the proof that $G \sim G(n,p)$ for $p = 1.1/n$, as the property we are interested in is clearly non-decreasing in p. Let $v \in V(G)$. We will show that the property in the lemma holds for v with probability $1 - o(n^{-1})$, so that the result follows by a simple union bound.

It may be useful to recall the following classical process to study the component of the sparse random graph G containing v. We begin by setting $X = \{v\}$, $X^- = \emptyset$ and $Z = V(G) \setminus \{v\}$. At each step, if possible, we choose an arbitrary vertex w from $X \setminus X^-$, move its neighbors in Z from Z into X and add w to X^-. It is well known, via comparison to an appropriate Galton-Watson process, that when the edge probability is c/n for any constant $c > 1$, then X becomes linear-sized (in n) with some probability bounded away from 0. For our purpose, we modify this process slightly to ensure that, even with an edge probability of $p = 1.1/n$, the set X remains small with probability $1 - o(n^{-1})$. Our key change is the introduction of an (initially empty) set Y, such that if w has exactly one neighbor not in X, which is not in Y, we instead move that neighbor from Z to Y. This adjustment uses the fact that the set we eventually find, A_v, for the application of Lemma 3.1, can have a non-empty neighborhood in G, so we only definitely wish to add a vertex to X if it has more than 1 neighbor in X^- or if it has a neighbor in X^- which has more than 1 neighbor outside of X. This is enough to drop the expected number of vertices we add to X at each stage below 1, so that the process will quickly terminate. We now carefully give this process in full.

We define A_v via a process, starting with $A_v = \{v\}$. In each step we add a (well-chosen) new vertex to A_v and reveal the edges of G incident to it. To aid us choosing the next vertex, we maintain sets X, Y, Z, X^-, such that throughout the whole process $X \cup Y \cup Z = V(G)$ is a partition of the vertex set, $X^- \subseteq X$, $\Delta(G[X^-, Y \cup Z]) \leq 1$, and moreover

$$|N(u) \cap X^-| = \begin{cases} 1 & \text{if } u \in Y \\ 0 & \text{if } u \in Z. \end{cases}$$

In the set X we will be collecting the vertices which eventually end up in A_v, and its subset X^- contains those vertices whose neighbors were already revealed by our process.

We index the sets X, X^-, Y, Z by the number i of the current step and initialize by defining $X_0 = \{v\}$, $X_0^- = Y_0 = \emptyset$ and $Z_0 = V(G) \setminus \{v\}$. Let $\sigma = 6000 \log n$. For $i \geq 0$, do the following:

1. If $i \leq \sigma$ and $|Z_i| < \frac{99n}{100}$ then terminate with failure.

2. If $X_i = X_i^-$ then set $A_v = X_i$ and terminate with success.

3. Pick an arbitrary $v_i \in X_i \setminus X_i^-$ and move it to X^-, that is, we define $X_{i+1}^- = X_i^- \cup \{v_i\}$.

4. Reveal the set $W_i = N(v_i) \cap (Y_i \cup Z_i)$ of neighbors of v_i outside of X_i.

    - If $W_i \cap Y_i = \emptyset$ and $|W_i \cap Z_i| \leq 1$ then we update our sets by $X_{i+1} = X_i$, $Y_{i+1} = Y_i \cup W_i$, and $Z_{i+1} = Z_i \setminus W_i$.

- Otherwise, we update by $X_{i+1} = X_i \cup W_i$, $Y_i = Y_i \setminus W_i$, and $Z_{i+1} = Z_i \setminus W_i$.

It is straightforward to check by induction that for these updates the conditions we promised are indeed maintained. In each round X^- grows exactly by one element, hence for every $i \geq 0$ we have $i = |X_i^-| \leq |X_i| \leq n$. Therefore, the procedure terminates after $t \leq n$ rounds. Our goal is to show that the procedure terminates with success after at most σ rounds with probability $1 - o(n^{-1})$. In the case we terminate with success after at most σ rounds, then $|A_v| = t \leq \sigma$, and since $\Delta(G[X_t^-, Y_t \cup Z_t]) \leq 1$, $X_t = X_t^-$ and $V(G) \setminus A_v = Y_t \cup Z_t$, we have $\Delta(G[A_v, V(G) \setminus A_v]) \leq 1$, as required in the statement of the lemma.

If $|Z_t| < \frac{99n}{100}$ and $t \leq \sigma$, then, among the at most tn possible edges we revealed in G we will have found at least $\frac{n}{100}$ edges. Thus,

$$\Pr\left(|Z_t| < \frac{99n}{100}\right) \leq \Pr\left(\mathrm{Bin}(\sigma n, p) \geq \frac{n}{100}\right) = o(n^{-2}),$$

and thus we terminate with failure with probability $o(n^{-1})$. Now, for each $0 \leq i < t$ let $n_i = |X_{i+1} \setminus X_i|$, and for each $i \geq t$ let $n_i = 0$. Then $|X_0| + \sum_{i=0}^{t-1} n_i = |X_t| = t$.

For each $i \geq 0$ and $j = 1, \ldots, n$, let b_i^j be independent Bernoulli variables with success probability p. For each $i \geq 0$ we use $b_i^1, \ldots, b_i^{|Y_i|+|Z_i|}$ to generate W_i and hence n_i. For each $i < t$, let $m_i = \sum_{j=1}^n b_i^j$, and for each $i \geq t$ let $m_i = 1$ if the process terminated with failure and otherwise let $m_i = \sum_{j=1}^n b_i^j$. Once we make it to step i, if we terminate either with failure or success then, respectively, the probability that $m_i - n_i \geq 1$ is 1 and $1 - (1-p)^n \geq 1 - e^{-1.1} > 1/3$. On the other hand, if we make it to step i and do not terminate with failure or success, then we have that the probability that $m_i - n_i \geq 1$ is at least the probability that $b_i^j = 1$ for exactly one i with $|Y_i| < i \leq |Z_i|$ and no i with $1 \leq i \leq |Y_i|$, which happens with probability $zp(1-p)^{y+z-1}$ where $z = |Z_i|$ and $y = |Y_i|$. As $z \geq \frac{99}{100}n$ because we did not terminate with failure, then this is at least $\frac{99n}{100}\frac{1.1}{n}(1-p)^n \geq 1.089 \cdot e^{-1.1 \times 1.01} > 1/3$, where we have used the inequality $1 - x \geq e^{-1.01x}$ which holds for every sufficiently small $x > 0$.

From the previous observation we have that the sequence of variables

$$S_i = \sum_{j=0}^{i} \min\{m_i - n_i - 1/3, 1\}$$

forms a submartingale with $|S_i - S_{i-1}| < 2$. By Azuma's inequality (Lemma 3.3) applied with $t = \sigma/12$, we conclude that with probability $1 - o(n^{-1})$

$$\sum_{i=0}^{\sigma-1}(m_i - n_i) \geq \frac{1}{4}\sigma. \tag{1}$$

Applying Lemma 3.2(ii) with $\mu = 1.1\sigma$ and $\delta = \frac{1}{44}$, with probability $1 - o(n^{-1})$ we have that

$$\sum_{i=0}^{\sigma-1}\sum_{j=1}^{n} b_i^j \leq \frac{9}{8}\sigma. \tag{2}$$

Thus, with probability $1 - o(n^{-1})$, we do not terminate with failure with probability $1 - o(n^{-1})$ and both (1) and (2) hold. As we did not terminate with failure, we have $\sum_{i=0}^{\sigma-1} m_i = \sum_{i=0}^{\sigma-1}\sum_{j=1}^{n} b_i^j$, so that, by summing (2) and (1), we have $|X_\sigma| = 1 + \sum_{i=0}^{\sigma-1} n_i \leq 1 + \frac{7}{8}\sigma$. However, for each $i \in [t]$, we have $|X_i| \geq i$, and thus $t \leq 1 + \frac{7}{8}\sigma \leq \sigma$. Altogether, then, we have shown that with probability $1 - o(n^{-1})$ we have $t \leq \sigma$, as required. □

We can now deduce Theorem 1.5, as follows.

Proof of Theorem 1.5. Let $G \sim G(n,p)$ for $p \leq \frac{1.1}{n}$. By Lemma 3.4, w.h.p., for each $v \in V(G)$ there is some $A_v \subset V(G)$ with $v \in A_v$ and $|A_v| \leq 6000 \log n$ such that $\Delta(G[A_v, V(G) \setminus A_v]) \leq 1$. Let $V' \subseteq V(G)$ be a subset of size $|V'| > 10^4 \log n$. Then, letting $v \in V'$ be arbitrary, we have that $\Delta(G[A_v \cap V', V' \setminus A_v]) \leq 1$, and, as $v \in A_v$, $|A_v| \leq 6000 \log n$ and $|V'| > 10^4 \log n$, that $A_v \cap V'$ is non-empty and that $|V' \setminus A_v| \geq 2$. Therefore, by Lemma 3.1, $G[V']$ is not globally rigid. □

4 Open problems

Random regular graphs. Following Benjamini and Tzalik [3], we studied the problem of global rigidity of random graphs in $\mathbb{R}$. A related natural question is, for which d is a random d-regular graph globally rigid in $\mathbb{R}$ with high probability? The methods developed in this article, in particular Lemma 2.1, together with known estimates on the second largest absolute eigenvalue of random d-regular graphs (e.g. see Friedman [5]) and the Expander Mixing Lemma, imply that a random d-regular graph is w.h.p. globally rigid in $\mathbb{R}$ for every $n \geq d \geq d_0$, where d_0 is a sufficiently large constant. The following argument shows $d_0 \geq 4$.

Theorem 4.1. *A 3-regular random graph is w.h.p. not globally rigid in* $\mathbb{R}$.

Proof. Let G be a random 3-regular graph, and let C be a shortest cycle of G. Let F_1 be the graph which consists of 2 triangles sharing one edge exactly, and F_2 be the graph which consists of a cycle $u_1u_2u_3u_4$ and a vertex x along with the edges xu_1 and xu_3. It is known that w.h.p. G does not contain any subgraph isomorphic to F_1 nor F_2 (see for instance Lemma 2.7 in the survey by Wormald [18]). If a vertex $b \in V(G) \setminus C$ would have two neighbors in C, then we would either obtain a shorter cycle or a copy of F_1 or F_2 – neither of which can happen. Therefore, applying Lemma 3.1 to $A = C$ and $B = V(G) \setminus C$ finishes the proof. □

Theorem 4.1 leaves open the problem of determining the smallest $d \geq 4$ for which a random d-regular graph with n vertices is globally rigid in $\mathbb{R}$ w.h.p., where we make the following conjecture.

Conjecture 4.2. *A random 4-regular graph with n vertices is globally rigid in $\mathbb{R}$ w.h.p.*

Algorithmic problem. We note that in the case of a fixed injective function f, Benjamini and Tzalik [3], as well as Girão, Illingworth, Michel, Powierski, and Scott [8], also considered the algorithmic problem of finding an injective function $f' : V(G) \to \mathbb{R}$ with $|f'(x) - f'(y)| = |f(x) - f(y)|$ for every edge $xy \in E(G)$ when G is a random graph. They obtain algorithms with polynomial expected running time. In our setup, where we generate only one random graph to reconstruct any injective function f, our proof does not provide any insight on how to find such f'. Note that there is a trivial algorithm with running time $O(|E(G)|2^n)$: take a BFS (or DFS) ordering of the vertices in G and then embed each next vertex both possible ways (either + or − the edge length to its parent vertex), and at the end check whether all pairwise distances are correct. We wonder whether this could be improved.

Problem 4.3. *Find an algorithm $\mathcal{A}$ with the following property: Let $G \sim G(n,p)$ for $p \gg \log n/n$. Then w.h.p. G is such that, for any injective $f\colon V(G) \to \mathbb{R}$, $\mathcal{A}(G,f)$ finds in polynomial time (depending only on n) an injective function f' satisfying $|f'(x) - f'(y)| = |f(x) - f(y)|$ for every edge $xy \in E(G)$.*

Higher dimensions. Finally, while one cannot hope for an extension of Theorem 1.2 to $\mathbb{R}^d$ for $d \geq 2$, it is conceivable that a statement of Theorem 1.3 is true for any $d \geq 2$. Even showing this for a given $f\colon [n] \to \mathbb{R}$ is an open problem, already suggested in [8], with some recent progress by Barnes, Petr, Portier, Randall Shaw, and Sergeev [1]. Here we state the global rigidity version.

Problem 4.4. *Show that, for every integer $d \geq 2$ and $\varepsilon > 0$, there exists $C > 0$ such that in the random graph $G \sim G(n, C/n)$ w.h.p. there exists a subset $V' \subseteq [n]$ of size $|V'| \geq (1-\varepsilon)n$ for which the induced subgraph $G[V']$ is globally rigid in $\mathbb{R}^d$.*

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