

Almost-full transversals in equi- n -squares

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Abstract

In 1975, Stein made a wide generalisation of the Ryser-Brualdi-Stein conjecture on transversals in Latin squares, conjecturing that every equi- n -square (an $n \times n$ array filled with n symbols where each symbol appears exactly n times) has a transversal of size $n - 1$. That is, it should have a collection of $n - 1$ entries that share no row, column, or symbol. In 2017, Aharoni, Berger, Kotlar, and Ziv showed that equi- n -squares always have a transversal with size at least $2n/3$. In 2019, Pokrovskiy and Sudakov disproved Stein's conjecture by constructing equi- n -squares without a transversal of size $n - \frac{\log n}{42}$, but asked whether Stein's conjecture is approximately true. I.e., does an equi- n -square always have a transversal with size $(1 - o(1))n$?

We answer this question in the positive. More specifically, we improve both known bounds, showing that there exist equi- n -squares with no transversal of size $n - \Omega(\sqrt{n})$ and that every equi- n -square contains $n - n^{1-\Omega(1)}$ disjoint transversals of size $n - n^{1-\Omega(1)}$.

1 Introduction

A *Latin square of order n* is an $n \times n$ array filled with n symbols where each symbol appears exactly once in each row and each column. A *transversal* (sometimes known as a *partial transversal*) is a collection of cells in the array which share no row, column, or symbol, while a *full transversal* in a Latin square of order n is a transversal with n cells. The *size* of a transversal is its number of cells. Latin squares, and transversals, have a long history of study dating back to Euler [5] who studied the decomposition of Latin squares into disjoint full transversals. As was known to Euler, a Latin square may have no full transversal. However, the prominent Ryser-Brualdi-Stein conjecture [13, 4], originating in the 1960's, suggests that every Latin square of order n should have a transversal with $n - 1$ cells and, moreover, one with n cells if n is odd. This conjecture has seen a lot of activity recently, culminating in the proof by the fourth author [8] that, when n is sufficiently large, every Latin square of order n has a transversal with $n - 1$ cells. For more on this, related results, and the history of the study of Latin squares, see the recent surveys [11, 9].

In 1975, Stein [14] made a series of bold conjectures that, broadly, suggest the Latin square conditions conjectured to guarantee a transversal of order $n - 1$ may be overkill. In particular, he conjectured that any equi- n -square has a transversal of size $n - 1$, where an equi- n -square is an $n \times n$ array filled with n symbols which each appear exactly n times. Thus, a Latin square of order n is an equi- n -square where we additionally require every symbol to appear at most once in each row or column. As some evidence towards his conjecture, Stein [14] used the probabilistic method to show that any equi- n -square contains a transversal of size at least $(1 - e^{-1})n$. This bound was the state of the art for some 40 years, until Aharoni, Berger, Kotlar, and Ziv [1] used topological methods to show that any equi- n -square contains a transversal of size at least $2n/3$. Very recently, Anastos and Morris [3] showed that any equi- n -square

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contains a transversal of size at least $(3/4 - o(1))n$. However, in 2019, Pokrovskiy and Sudakov [12] had shown that Stein's conjecture is, indeed, over-ambitious. That is, they constructed equi- n -squares that have no transversal with size larger than $n - \frac{1}{42} \log n$.

While this settles the falsity of Stein's conjecture, the bounds $n - O(\log n)$ and $(3/4 - o(1))n$ on the size of the largest transversal that can be guaranteed in any equi- n -square are rather far apart. In particular, Pokrovskiy and Sudakov [12] (see also [11, Problem 4.3]) asked whether Stein's conjecture holds asymptotically, i.e., does an equi- n -square always have a transversal with $(1 - o(1))n$ cells? Our main result is to confirm that it not only does, but that it can moreover be almost decomposed into such transversals, with the following result.

Theorem 1.1. *There exists $\varepsilon > 0$ such that every equi- n -square contains at least $n - n^{1-\varepsilon}$ disjoint transversals with size at least $n - n^{1-\varepsilon}$.*

We also modify Pokrovskiy and Sudakov's construction from [12] to show that a related but simpler construction provides equi- n -squares that must omit far more symbols in any transversal, as follows.

Theorem 1.2. *For each $n \in \mathbb{N}$, there is an equi- n -square with no transversal of size $n - \left(\frac{1}{2\sqrt{2}} + o(1)\right) \sqrt{n}$.*

As our construction is very natural, it is reasonable to suggest that the constant $\frac{1}{2\sqrt{2}}$ in Theorem 1.2 is unimprovable. With slightly less ambition, we suggest the following to replace Stein's disproved conjecture.

Conjecture 1.3. $\exists C > 0$ *such that every equi- n -square has a transversal with size at least $n - C\sqrt{n}$.*

As is well-known, a Latin square L of order n has a natural representation as an n -regular linear 3-partite 3-uniform $(3n)$ -vertex hypergraph $\mathcal{H}(L)$, where 3 vertex classes of size n represent respectively the rows, columns, and symbols of L and a 3-uniform edge of the row, column, and symbol is added to $\mathcal{H}(L)$ for each cell in L . As L is an equi- n -square, $\mathcal{H}(L)$ is n -regular, and as L is a Latin square each codegree in $\mathcal{H}(L)$ is at most 1 (i.e., $\mathcal{H}(L)$ is linear). From this perspective, the main question we have considered asks: in an n -regular 3-partite 3-uniform $(3n)$ -vertex hypergraph \mathcal{H} with vertex classes A, B and C of size n , if the codegree of x and y is at most 1 for each $x \in A$ and $y \in B$, then does \mathcal{H} have a transversal of size $(1 - o(1))n$ (or, indeed, an almost decomposition into such transversals)? Thus, for Theorem 1.1, we dispense with 2/3rds of the codegree conditions for the Latin square case. Note that we cannot dispense with all of the codegree conditions here, due to the following construction adapted from an example of Alon and Kim [2].

Let $t \in \mathbb{N}$, $X = \{x_1, \dots, x_{2t}, x'_1, \dots, x'_t\}$, $Y = \{y_1, \dots, y_{2t}, y'_1, \dots, y'_t\}$ and $Z = \{z_1, \dots, z_{2t}, z'_1, \dots, z'_t\}$. Let $V(\mathcal{H}) = X \cup Y \cup Z$ and

$$E(\mathcal{H}) = \{(x_i, y_i, z'_j), (x_i, y'_j, z_i), (x'_j, y_i, z_i) : i \in [2t], j \in [t]\}.$$

Note that any matching in \mathcal{H} can contain at most 1 edge intersecting $\{x_i, y_i, z_i\}$ for each $i \in [2t]$, and therefore any matching in \mathcal{H} can have at most $2t$ edges, and hence cover at most $6t = 2|\mathcal{H}|/3$ vertices. Moreover, \mathcal{H} is $2t$ -regular and 3-partite. As \mathcal{H} has $n := 3t$ vertices in each class, this does not quite correspond to our case, but blowing up each vertex in \mathcal{H} by 3 gives an $18t$ -regular 3-partite 3-graph in which any matching must omit at least 1/9 of the vertices and in which there are $3n = 9t$ vertices in each class. Then, taking 2 disjoint copies of this graph gives an $18t$ -regular 3-partite 3-graph with $n' = 3 \cdot 18t$ vertices in which any perfect matching must omit at least $n'/9$ vertices.

Later, in Section 4.2, we will deduce Theorem 1.1 from a more general result, Theorem 4.3. Similarly, we could deduce a version of Theorem 1.1 with further weakened codegree conditions to say that (using the notation above) $\text{cod}_{\mathcal{H}}(x, y) \leq n^{1-\mu}$, as long as $\varepsilon \ll \mu$. Here, the corresponding square has n copies of each symbol, and n symbols in each row and column, but we allow up to $n^{1-\mu}$ symbols to appear in any one square as long as they are all different.

In Section 2, we prove Theorem 1.2. In Section 3, we sketch a simplified version of our methods for Theorem 1.1, showing how to find a large transversal in certain equi- n -squares. In Section 4, we then prove Theorem 1.1.

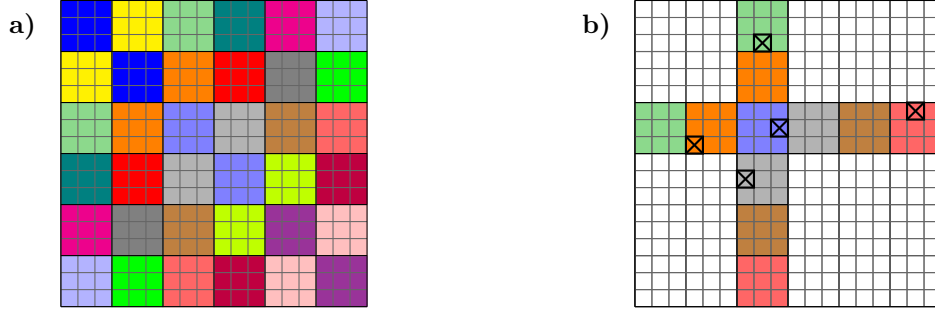


Figure 1: a) Our construction in the simplified case when $n = 2m^2$ with $m = 3$. b) The boxes comprising C_3 in the same equi- n -square with a transversal highlighted – if a light blue square appears in C_3 in a transversal then at most 4 of the 5 colours appearing only in C_3 can appear. Thus, at least one colour from $C_3 \cup C_4$ is omitted in any transversal.

2 Equi- n -squares with no partial transversal of size $n - \Omega(\sqrt{n})$

To illustrate the main ideas of our construction, we first prove Theorem 1.2 whenever $n = 2m^2$ for some integer m , constructing equi- n -squares with no transversal of size larger than $n - \frac{1}{2\sqrt{2}}\sqrt{n}$ as depicted in Figure 1a). We will then show how this construction can be modified for general n without changing the size of the largest possible transversal by much. In our construction we use a key part of the construction by Pokrovskiy and Sudakov [12] of equi- n -squares with no transversal of size more than $n - \frac{1}{42} \log n$, which, roughly, is the argument depicted in Figure 1b).

Proof of Theorem 1.2 whenever $n = 2m^2$ for some integer m . We construct an equi- n -square S with both rows and columns indexed by elements in $[n]$, where $n = 2m^2$ for some $m \in \mathbb{Z}$. We use the notation (x, y) to denote the cell in the x -th row and y -th column. We partition S into $(2m)^2$ boxes, each containing m^2 cells, as follows. For every $i, j \in [2m]$, let

$$A_{i,j} = \{(x, y) : 1 + (i - 1)m \leq x \leq im \text{ and } 1 + (j - 1)m \leq y \leq jm\}.$$

We now pair up the boxes and assign a unique colour to each pair, noting that this uses $\frac{(2m)^2}{2} = 2m^2 = n$ colours, as depicted in Figure 1a). For each $k \in [m]$, we pair the boxes $A_{2k-1, 2k-1}$ and $A_{2k, 2k}$ and colour every cell they contain the same colour. For each $1 \leq i < j \leq 2m$, we pair the boxes $A_{i,j}$ and $A_{j,i}$ and colour every cell they contain the same colour. Notice that as each colour appears in exactly $2m^2 = n$ many cells, we have constructed an equi- n -square.

Letting S be the equi- n -square we have constructed, we now show it has no transversal with more than $n - m/2$ cells. Suppose, then, that T is a transversal of S . For each $k \in [2m]$, let

$$C_k = \left(\bigcup_{i \in [2m]} A_{i,k} \right) \cup \left(\bigcup_{j \in [2m]} A_{k,j} \right),$$

noting this is the disjoint union of $4m - 1$ boxes. As depicted in Figure 1b), we now show the following claim.

Claim 2.1. For every $k \in [m]$, at least one colour used in $C_{2k-1} \cup C_{2k}$ is not used in T .

Proof of Claim 2.1. Let $k \in [m]$, and suppose that the colour in boxes $A_{2k-1, 2k-1}$ and $A_{2k, 2k}$ is blue. If T misses out on the colour blue, we would be done. Suppose then that T contains a blue cell in $A_{j,j}$ for $j \in \{2k - 1, 2k\}$. As $\bigcup_{i \in [2m]} A_{i,j} \subset C_j$ contains elements only in m columns of S , T has at most m cells in $\bigcup_{i \in [2m]} A_{i,j}$. Similarly, T has at most m cells in $\bigcup_{i \in [2m]} A_{j,i} \subset C_j$. As T has a cell in $A_{j,j}$, it therefore has at most $2m - 1$ cells in C_j , and thus at most $2m - 2$ cells in C_j which are not blue. However, C_j uses $2m - 1$ non-blue colours, each of which do not appear in $S \setminus C_j$. Thus, T misses out on at least one colour used on $C_j \subset C_{2k-1} \cup C_{2k}$. \square

To finish, observe that each colour appears on at most two of $C_{2k-1} \cup C_{2k}$, $k \in [m]$. This combined with Claim 2.1 implies that T must miss out on at least $\frac{m}{2}$ colours, and therefore $|T| \leq n - \frac{m}{2} = n - \frac{\sqrt{n}}{2\sqrt{2}}$. \square

We now deal with the general n case. Above we considered boxes of equal side lengths and now we will allow the side lengths to differ. We will choose these lengths so that they are close to each other and that the number of cells in each box is as close to $n/2$ as possible, though in general slightly below this. We use a similar construction to the special case, but each colour may need to be used in $O(n^{1/4})$ cells outside of its pair of boxes to make up n cells of that colour. These $O(n \cdot n^{1/4})$ cells will be grouped together into $O(n^{1/4})$ rows and columns, so that any transversal would contain at most $O(n^{1/4})$ of them, which will not affect significantly the argument given above for the maximum size of a transversal.

Proof of Theorem 1.2. Once again, we will index the rows and columns of the equi- n -square by elements of $[n]$. We will use rectangular boxes and so now choose the side lengths, denoted a and b . For this let $m = \lceil \sqrt{n/2} \rceil$, $r = \lceil \sqrt{m^2 - n/2} \rceil$ and take $a = m + r$, $b = m - r$. We first note that $0 \leq m^2 - n/2 \leq 2m \leq 4\sqrt{n/2}$, and then, similarly, $0 \leq r^2 - (m^2 - n/2) \leq 2r \leq 4\sqrt{4\sqrt{n/2}} \leq 8n^{1/4}$. Consequently, as $n - 2ab = 2(n/2 - (m^2 - r^2))$,

$$n - 16n^{1/4} \leq 2ab \leq n$$

Moreover, $b \geq \sqrt{n/2} - r > \sqrt{n/2} - 8n^{1/4}$.

For each $i \in [2a]$ and $j \in [2b]$ take the box

$$B_{i,j} = \{(x, y) : 1 + (i-1)b \leq x \leq ib \text{ and } 1 + (j-1)a \leq y \leq ja\}.$$

As before, we pair up the boxes and assign a unique colour to each pair, using that there will be $2ab \leq n$ pairs of boxes, so that each pair can have its own colour. Note that we must use $n - 2ab \leq 16n^{1/4}$ entries of each colour elsewhere in the equi- n -square. Noting also that $b \leq a$, we assign the same colour to $B_{2k-1, 2k-1}$ and $B_{2k, 2k}$ for each $k \in [b]$. We assign the same colour to $B_{i,j}$ and $B_{j,i}$ for each $1 \leq i < j \leq b$. Additionally, for each $b+1 \leq s \leq a$ and $t \in [2b]$, we assign the same colour to $B_{2s-1, t}$ and $B_{2s, t}$. Finally, we use the leftover of the colours used as well as the other colours we have not yet used to arbitrarily colour the cells which have not yet been coloured. This finishes the construction of our equi- n -square.

Letting S be the constructed square, we now show it has no transversal with size $n - \left(\frac{1}{2\sqrt{2}} + o(1)\right)\sqrt{n}$. Suppose then that T is a transversal of S . Let S' be the sub-square consisting of all the cells in the first $2ab$ rows and $2ab$ columns of S , and let $T' = T \cap S'$. Note that $|T \cap (S \setminus S')| \leq 2(n - 2ab) \leq 32n^{1/4}$. We now solely focus on T' and use an almost identical argument to the one in the special case done before. For each $k \in [2b]$, let

$$C_k := \left(\bigcup_{i \in [2a]} B_{i,k} \right) \cup \left(\bigcup_{j \in [2b]} B_{k,j} \right).$$

Note that every C_k is the disjoint union of $2a + 2b - 1$ boxes. We now show the corresponding version of Claim 2.1.

Claim 2.2. For every $k \in [b]$, at least one colour used in $C_{2k-1} \cup C_{2k}$ is not used in T' .

Proof of Claim 2.2. Suppose that the colour in boxes $B_{2k-1, 2k-1}$ and $B_{2k, 2k}$ is blue. If T' has no blue cell, we would be done. Hence we assume that for some $j \in \{2k-1, 2k\}$, T' contains a cell in $B_{j,j}$. As $\bigcup_{i \in [2a]} B_{i,j} \subset C_j$ contains elements in exactly a columns of S' , T' has at most a cells in $\bigcup_{i \in [2a]} B_{i,j}$. Similarly, T' has at most b cells in $\bigcup_{i \in [2b]} B_{j,i}$. As T' has a cell in $B_{j,j}$, it therefore has at most $a+b-1$ cells in C_j , and thus at most $a+b-2$ cells in C_j which are not blue. However, C_j uses $(1/2)(2a-1+2b-1) = a+b-1$ non-blue colours, each of which does not appear in $S' \setminus C_j$. Thus, T' misses out on at least one colour used on $C_j \subset C_{2k-1} \cup C_{2k}$. \square

Observe that, within S' , any colour appears in at most two of the sets $C_{2k-1} \cup C_{2k}$ for different $k \in [b]$. This combined with Claim 2.2 implies that T' must miss out on at least $b/2$ colours. Thus,

$$|T| = |T'| + |T \cap (S \setminus S')| \leq n - \frac{b}{2} + 32n^{1/4} \leq n - \frac{\sqrt{n}}{2\sqrt{2}} + 36n^{1/4} = n - (1 - o(1))\frac{\sqrt{n}}{2\sqrt{2}}. \quad \square$$

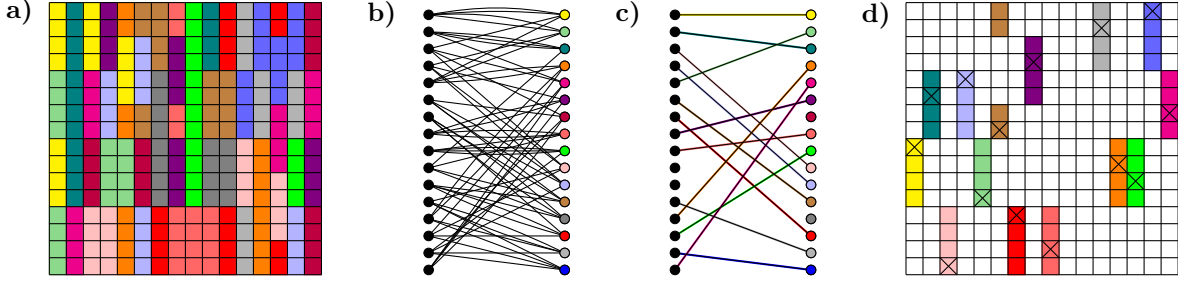


Figure 2: **a)** A subsquare S in which each column can be decomposed into 4 blocks with size 4 which have the same colour (for ease of visualisation, cells of each colour are often drawn next to each other in natural blocks, but this is not required in general). **b)** The multigraph K corresponding to S . **c)** A matching M in K . **d)** The subsquare $S' \subset S$ corresponding to the matching M . Any set of cells which share no row or column must then share no colour and hence be a transversal; an example is marked by crosses.

3 Proof sketch: one large transversal in certain equi- n -squares

Suppose S is an equi- n -square. When each symbol appears $o(n)$ times in each row and column, a large transversal is known to exist via the Rödl nibble, where this large transversal satisfies certain pseudorandom conditions. Thus, it is important to consider equi- n -squares in which symbols appear many times in the same row or the same column – to discuss our methods we will choose a special case where each symbol is repeated many times in a small number of columns. Suppose that $n = 4m$ with $m \in \mathbb{N}$, and that the cells of S can be partitioned into a set of blocks \mathcal{B} , where each $B \in \mathcal{B}$ contains m cells which are all in the same column and all have the same symbol (see, for example, Figure 2). Using Hall’s matching criterion it is not hard to show that the columns of S can be perfectly matched into \mathcal{B} with a matching M such that each column is matched to a block in that column, and every block in this matching has a different symbol. Let S' be the subsquare of S formed by retaining only those cells whose block appears in an edge of M , and note that there are m entries in each column of S' . Our aim (at the expense of finding only an *almost*-perfect matching) is to find a random such matching M for which it is likely that almost every row has around m entries in S' . Using (a defect version of) Hall’s matching criterion in this roughly regular S' , it is not hard then to find a collection of $(1 - o(1))n$ cells in S' which share no row or column, whereupon the definition of S' implies they share no symbol. Thus, we will have found a large transversal in S (see, again, Figure 2). We will find our random matching M using a novel ‘bounded-dependence matching algorithm’.

Before discussing this in more detail, along with how we find an appropriate random matching, let us note that at a very high level this is inspired by work of the second author with Pokrovskiy and Sudakov on Ringel’s conjecture [10]. That is to say, when the codegrees are small, we can find a large transversal using semi-random methods. Where many colours appear in the same row or the same column, some indication of which colour we might wish to use for each column could be selected deterministically using some matching like the initial (non-random) matching M . Key (as in [10]) is to find an appropriate way to randomise this deterministic selection so that it can be used along with the semi-random method. Indeed, though proving Theorem 1.1 in full will involve several more complications, this sketch encompasses the main novelty in our methods, which can then be developed into a more complex scheme using the semi-random method, as described and done in Section 4.

Let us return to our equi- n -square S and the decomposition \mathcal{B} of S into blocks of size m . Let \mathcal{J} be the set of columns and let \mathcal{A} be the set of symbols. Let K be an auxiliary bipartite multigraph with vertex classes \mathcal{J} and \mathcal{A} , where, for each $B \in \mathcal{B}$, we add an edge labelled with B between the column and the symbol of B . Note that K is a 4-regular bipartite multigraph. Therefore, using Hall’s matching criterion, it can be decomposed into 4 matchings, say M_1, M_2, M_3, M_4 .

We will use M_1, M_2, M_3 , and M_4 to generate our random matching M , by finding random matchings $M'_1 \subset M_1 \cup M_2$ and $M'_2 \subset M_3 \cup M_4$, and then selecting $M \subset M'_1 \cup M'_2$ randomly. For example, for M'_1 the initial idea is to observe that $M_1 \cup M_2$ is a union of cycles and, for each such cycle, randomly choose the

‘odd or even’ edges of the cycle and add them to M'_1 . This creates a matching, but if the cycles are long then there will be a lot of dependence between whether different edges appear in M'_1 , or not, preventing us from showing that the number of entries in each row in blocks in M'_1 is suitably concentrated. The key idea then is to take $M_1 \cup M_2$ and delete a few edges to break the cycles into paths/cycles which are each not too long. Doing similarly with $M_3 \cup M_4$ to generate M'_2 , we then similarly generate M from $M'_1 \cup M'_2$ (which is the disjoint union of paths/cycles rather than just cycles). By choosing the maximum length of the paths/cycles allowed after the edge deletion carefully, we can (as described below) ensure that M is a large (almost-perfect) matching while limiting the dependence over the appearance of different edges in M with each other. We then track certain Lipschitz random variables and show that they are likely to be concentrated.

More specifically, let $k = n^{1/3} \log^{-2} n$. Let S_1, \dots, S_r be a minimal set of vertex-disjoint paths/cycles in $M_1 \cup M_2$, each with length at most k , the union of whose vertex sets is $V(M_1 \cup M_2)$. Note that $|E(M_1 \cup M_2) \setminus E(S_1 \cup \dots \cup S_r)| \leq 2n/k$. For each $i \in [r]$, properly edge colour S_1, \dots, S_r with red and blue (noting all the cycles are even). For each $i \in [r]$, choose x_i uniformly and independently at random from $\{0, 1\}$. Let M'_1 be the union across $i \in [r]$ of the red edges from $E(S_i)$ with $x_i = 0$ and the blue edges of $E(S_i)$ with $x_i = 1$. Observe that M'_1 is a matching. Similarly, use M_3 and M_4 to randomly generate M'_2 . Then, similarly use $M'_1 \cup M'_2$ to generate M .

We use M to tell us from which column we will select a cell with which symbol in it (and which symbols we will not select a cell for, but these will be a small number of symbols overall). For this, let \mathcal{I} be the set of n rows. Form the auxiliary bipartite graph L on vertex classes \mathcal{I} and \mathcal{J} , where there is an edge ij exactly if j is matched to a symbol with an edge in M corresponding to a block containing the cell (i, j) . Observe, as M is a matching, that a matching in L corresponds to a transversal in S .

For each row $i \in \mathcal{I}$, let $m_1(i)$ be the number of edges $jc \in M_1$ which correspond to a block containing a cell on row i . Define $m_2(i), m_3(i), m_4(i), m'_1(i), m'_2(i)$ and $m(i)$ similarly. Note that $m_1(i) + m_2(i) + m_3(i) + m_4(i) = n$ and $\mathbb{E}(m'_1(i)) \leq (m_1(i) + m_2(i))/2$. As $|E(M_1 \cup M_2) \setminus E(S_1 \cup \dots \cup S_r)| \leq 2n/k$, we have

$$\mathbb{E}(m'_1(i)) \geq \frac{m_1(i) + m_2(i)}{2} - \frac{2n}{k} \geq \frac{m_1(i) + m_2(i)}{2} - n^{2/3} \log^{O(1)} n.$$

Crucially, changing the value of any one of $x_i, i \in [r]$, changes the value of $m'_1(i)$ by at most $k = n^{1/3} \log^{-2} n$. Thus, by an application of McDiarmid’s inequality (Lemma 4.4), we get that, with probability $1 - o(n^{-1})$,

$$m'_1(i) = \frac{m_1(i) + m_2(i)}{2} \pm \left(n^{2/3} \log^{O(1)} n + O\left(\sqrt{nk \log n}\right) \right) = \frac{m_1(i) + m_2(i)}{2} \pm n^{2/3} \log^{O(1)} n.$$

Similarly, with probability $1 - o(n^{-1})$, $m'_2(i) = \frac{m_3(i) + m_4(i)}{2} \pm n^{2/3} \log^{O(1)} n$. Similarly, then, with probability $1 - o(n^{-1})$,

$$d_L(i) = m(i) = \frac{m_1(i) + m_2(i) + m_3(i) + m_4(i)}{4} \pm n^{2/3} \log^{O(1)} n = \frac{n}{4} \pm n^{2/3} \log^{O(1)} n = m \pm n^{2/3} \log^{O(1)} n.$$

For each column $j \in \mathcal{J}$ with j in some edge in M , we have $d_L(j) = m = n/4$, and otherwise $d_L(j) = 0$. Then, for each $U \subset \mathcal{I}$, we have $(m - n^{2/3} \log^{O(1)} n) \cdot |U| \leq m \cdot |N_L(U)|$, so that

$$|N_L(U)| \geq |U| \left(1 - \frac{n^{2/3} \log^{O(1)} n}{m} \right) \geq |U| - n^{2/3} \log^{O(1)} n.$$

Thus, by a defect version of Hall’s matching theorem, L contains a matching with size $n - n^{2/3} \log^{O(1)} n$, and hence S has a transversal with $n - n^{2/3} \log^{O(1)} n$ cells.

4 Almost decomposing equi- n -squares

We will use the following result of Molloy and Reed [7].

Theorem 4.1. *For all k there is a constant C_k such that any k -uniform hypergraph of maximum codegree B and maximum degree Δ has list chromatic index at most $(1 + C_k(B/\Delta)^{1/k}(\log(\Delta/B))^4) \Delta$.*

Specifically, we will use the following corollary.

Corollary 4.2. *For all $\varepsilon > 0$ there is a $\mu > 0$ and D_0 such that the following holds for each $D \geq D_0$. Suppose \mathcal{H} is a 3-uniform hypergraph with maximum degree at most D such that, for each distinct $x, y \in V(\mathcal{H})$, either $\text{cod}_{\mathcal{H}}(x, y) \leq D^{1-\varepsilon}$ or every edge contains either both x and y or none of them.*

Then, \mathcal{H} has chromatic index at most $(1 + D^{-\mu})D$.

Proof. Let $0 < 1/D_0 \ll \mu \ll \varepsilon$. Let $E = \{xy : \text{cod}_{\mathcal{H}}(x, y) > D^{1-\varepsilon}\}$, and note that E is a matching. For each $f \in E$, pick $x_f \in V(f)$. For each $f \in E$ and $e \in E(\mathcal{H})$ with $V(f) \subset V(e)$ create a new vertex, v_e . Let \mathcal{H}' be the hypergraph with vertex set $(V(\mathcal{H}) \setminus \{x_f : f \in E\}) \cup (\cup_{f \in E} \cup_{e \in E(\mathcal{H})} v_e)$ and edge set

$$(E(\mathcal{H}) \setminus (\cup_{f \in E} \cup_{e \in E(\mathcal{H}): V(e) \subset V(f)} e)) \cup (\cup_{f \in E} \cup_{e \in E(\mathcal{H}): V(e) \subset V(f)} (\{v_e\} \cup (V(e) \setminus \{x_f\}))).$$

Observe that $\Delta(\mathcal{H}') \leq D$ and $\text{cod}_{\mathcal{H}'}(x, y) \leq D^{1-\varepsilon}$ for each $x, y \in V(\mathcal{H}')$. Then, by Theorem 4.1, we have $\chi'(\mathcal{H}') \leq (1 + D^{-\mu})D$. Let χ' be a colouring of $E(\mathcal{H}')$ with at most $(1 + D^{-\mu})D$ colours so that the edges of each colour forms a matching. For each $e \in E(\mathcal{H}') \cap E(\mathcal{H})$ let $\chi(e) = \chi'(e)$. For each $e \in E(\mathcal{H}) \setminus E(\mathcal{H}')$, take $f \in E$ such that $V(f) \subset V(e)$ and let $\chi(e) = \chi'((V(e) \setminus \{x_f\}) \cup \{v_e\})$. Observing that χ is a colouring of $E(\mathcal{H})$ into at most $(1 + D^{-\mu})D$ colours in which the edges of each colour form a matching, completes the proof. \square

We will use this with our bounded-dependence random matching algorithm to prove the following.

Theorem 4.3. *There are some $\xi, \eta > 0$ and D_0 such that the following holds for each $D \geq D_0$. Let \mathcal{H} be a 3-uniform hypergraph with maximum degree D and at least \sqrt{D} vertices. Suppose that the graph on $V(\mathcal{H})$ with edges xy present if $\text{cod}_{\mathcal{H}}(x, y) \geq D^{1-\eta}$ is bipartite.*

Then, there is some $\mathcal{H}' \subset \mathcal{H}$ with $e(\mathcal{H}') \geq e(\mathcal{H}) - |\mathcal{H}| \cdot D^{1-\xi}$ and

$$\chi'(\mathcal{H}') \leq (1 + D^{-\xi})D.$$

We will then deduce Theorem 1.1 from this. We will use McDiarmid's inequality, in the following form (see [6, Lemma 1.2]).

Lemma 4.4. *Let $n \in \mathbb{N}$ and $c_1, \dots, c_n \geq 0$. For each $i \in [n]$, let X_i be an independent random variable taking values in Ω_i , and let $X = (X_1, \dots, X_n)$. Let $f : \prod_{i=1}^n \Omega_i \rightarrow \mathbb{R}$ be a function such that, for each $i \in [n]$, changing X in the i th co-ordinate changes the value of $f(X)$ by at most c_i .*

Then, for all $t > 0$,

$$\mathbb{P}(|f(X) - \mathbb{E}(f(X))| > t) \leq 2 \exp\left(-\frac{t^2}{\sum_{i \in [n]} c_i^2}\right).$$

4.1 A random subhypergraph with 'all or small' codegrees

To prove Theorem 4.3, we will first use our methods to prove the following lemma.

Lemma 4.5. *There are some $\varepsilon, \mu, \eta > 0$ and D_0 such that the following holds for each $D \geq D_0$. Let \mathcal{H} be a 3-uniform hypergraph with maximum degree D and at least \sqrt{D} vertices. Suppose that the graph on $V(\mathcal{H})$ with edges xy present if $\text{cod}_{\mathcal{H}}(x, y) \geq D^{1-\eta}$ is bipartite.*

Then, there is some $D^{3\mu} \leq t \leq 8D^{3\mu}$ and a random subhypergraph $\mathcal{H}' \subset \mathcal{H}$ such that

A1 *for each $e \in E(\mathcal{H})$, $\mathbb{P}(e \in E(\mathcal{H}')) \leq 1/t$,*

A2 $\Delta(\mathcal{H}') \leq (1 + D^{-\varepsilon})D/t$,

A3 *for each distinct $x, y \in V(\mathcal{H})$, either $\text{cod}_{\mathcal{H}'}(x, y) \leq D^{1-\varepsilon}/t$ or every $e \in E(\mathcal{H}')$ satisfies $\{x, y\} \subset V(e)$ or $\{x, y\} \cap V(e) = \emptyset$,*

and, with probability at least $1 - D^{-10}$, the following holds.

A4 $e(\mathcal{H}') \geq e(\mathcal{H})/t - D^{1-\varepsilon}|\mathcal{H}|/t$.

Proof. Let $0 < 1/D_0 \ll \varepsilon \ll \gamma \ll \mu \ll \eta \ll 1$ and $D \geq D_0$. (Probably, we can simply take $\eta = 2\mu$.) Let μ be such that $k = D^\mu$ is a power of 2, let $\ell \in \mathbb{N}$ then be such that $k = 2^\ell$, and let $t = k^3$. Let $V = V(\mathcal{H})$ and $m = \lfloor D/k \rfloor$. As set-up, do the following.

- B1** Remove from \mathcal{H} any edge xyz in which $\{x, y\}$ and $\{y, z\}$ both have codegree at least m , and let the result be \mathcal{H}_0 .
- B2** Take a maximal collection \mathcal{B} of disjoint subsets of $E(\mathcal{H}_0)$ with size m such that, for every $B \in \mathcal{B}$, there is some set $U_B \subset V$ with $|U_B| = 2$ and $U_B \subset V(e)$ for each $e \in B$.
- B3** Let $\mathcal{B}_v, v \in V$, maximise $\sum_{v \in V} |\mathcal{B}_v|$ subject to the following conditions.
 - B3.1** For each $v \in V$, the sets in \mathcal{B}_v are disjoint subsets of $\{e \in E(\mathcal{H}_0) : v \in V(e)\} \setminus (\cup_{B \in \mathcal{B}: v \in U_B} B)$ with size m .
 - B3.2** For each distinct $v, w \in V$ and $B \in \mathcal{B}_v$, the number of edges in B containing w is at most $D^{2\gamma}m/k$.
 - B3.3** For each distinct $v, w \in V$ and $B \in \mathcal{B}_v, B' \in \mathcal{B}_w, |B \cap B'| \leq D^\gamma m/k^2$.

Let \mathcal{H}_1 be the subhypergraph of \mathcal{H}_0 which contains exactly the edges $e = uvw \in E(\mathcal{H}_0)$ for which either there is some $B \in \mathcal{B}$ with $e \in B$ and some $v \in V$ and $B' \in \mathcal{B}_v$ with $e \in B'$ or, for each $v \in V(e)$, there is some $B \in \mathcal{B}_v$ with $e \in B$. For each $B \in \mathcal{B}$, let $A_B^0 = B$.

Let K be an auxiliary multigraph with vertex set $V \cup (\cup_{v \in V} \mathcal{B}_v)$ where, for each $B \in \mathcal{B}$, we add U_B to $E(K)$ and, for each $v \in V$ and $B \in \mathcal{B}_v$, we add vB to $E(K)$. Note that, by the conditions on \mathcal{H} , we have that K is a bipartite multigraph with maximum degree at most k , and therefore (for example, by embedding K into a k -regular bipartite multigraph and repeatedly using Hall's matching criterion to find a perfect matching) $\chi'(K) \leq k$. Thus, we can partition K into matchings M_1^0, \dots, M_k^0 . Let $s = D^{\gamma/2}$.

For each $1 \leq h \leq \ell$ in turn do the following with the matchings $M_1^{h-1}, \dots, M_{k/2^{h-1}}^{h-1}$ to generate matchings $M_1^h, \dots, M_{k/2^h}^h$ randomly:

- Let $F_h \subset M_1^{h-1} \cup \dots \cup M_{k/2^{h-1}}^{h-1}$ be a minimal set of edges such that, for each $i \in [k/2^h]$, if, for the appropriate $r_i^h, M_{2i}^{h-1} \cup M_{2i-1}^{h-1} - F_h$ (considered as a multigraph and removing any copies of edges in F_h) has as its components the paths/cycles $S_{i,1}^h, \dots, S_{i,r_i^h}^h$, then, each such path/cycle has length at most s .
- For each $i \in [k/2^h]$, and each $j \in [r_i^h]$, pick $x_{i,j}^h$ uniformly and independently at random from $\{0, 1\}$ and let
$$M_i^h = \left(\cup_{j \in [r_i^h]: x_{i,j}^h = 1} E(S_{i,j}^h) \cap M_{2i}^{h-1} \right) \cup \left(\cup_{j \in [r_i^h]: x_{i,j}^h = 0} E(S_{i,j}^h) \cap M_{2i-1}^{h-1} \right).$$
- For each $B \in \mathcal{B}$, let $A_B^h \subset A_B^{h-1}$ be chosen by including each element of A_B^{h-1} independently at random with probability $1/2$.

Let $M = M_1^\ell$ and $\mathcal{H}_1^0 = \mathcal{H}_2^0 = \mathcal{H}_1$. For each $h \in [\ell]$, let $M^h = \cup_{i \in [k/2^h]} M_i^h$. For each $h \in [\ell]$, let \mathcal{H}_0^h be a subhypergraph of \mathcal{H}_2^{h-1} which maximises $e(\mathcal{H}_0^h)$ subject to the following conditions.

- C1** For each $B \in \mathcal{B}$ and each $v \in V$ and $B' \in \mathcal{B}_v$, if $(B \cap B') \cap E(\mathcal{H}_0^h) \neq \emptyset$, then $U_B, vB' \notin F^h$ and U_B and vB' do not appear in any path/cycle in $S_{i,1}^h, \dots, S_{i,r_i^h}^h$ together, for any $i \in [k/2^h]$.
- C2** For each distinct $v, w \in V$ and $B \in \mathcal{B}_v, B' \in \mathcal{B}_w$, if $(B \cap B') \cap E(\mathcal{H}_0^h) \neq \emptyset$, then $vB, wB' \notin F^h$ and vB and wB' do not appear in any path/cycle in $S_{i,1}^h, \dots, S_{i,r_i^h}^h$ together, for any $i \in [k/2^h]$.

Then, for each $h \in [\ell]$, let \mathcal{H}_1^h be the hypergraph with vertex set V and edges $e \in E(\mathcal{H}_0^h)$ for which either there is some $B \in \mathcal{B}$ with $e \in B$ and some $v \in V$ and $B' \in \mathcal{B}_v$ with $e \in B'$, where $U_B, vB' \in M^h$ and $e \in A_B^h$, or, for each $v \in V(e)$, there is some $B \in \mathcal{B}_v$ with $e \in B$ and $vB \in M^h$.

Then, for each $h \in [\ell]$, let \mathcal{H}_2^h be a subhypergraph of \mathcal{H}_1^h which maximises $e(\mathcal{H}_2^h)$ subject to the following conditions.

D1 For each $B \in \mathcal{B} \cup (\cup_{v \in V} \mathcal{B}_v)$, $|B \cap E(\mathcal{H}_2^h)| \leq (1 + h \cdot D^{-\varepsilon}/\ell)m/2^{2h}$.

D2 For each distinct $v, w \in V$ and $B \in \mathcal{B}_v$, the number of edges in $B \cap E(\mathcal{H}_2^h)$ containing w is at most $(1 + h/\ell)D^{2\gamma}m/2^{2h}k$.

D3 For each distinct $v, w \in V$ and $B \in \mathcal{B}_v, B' \in \mathcal{B}_w$, $|(B \cap B') \cap E(\mathcal{H}_2^h)| \leq (1 + h/\ell)D^\gamma m/2^h k^2$.

Finally, let $\mathcal{H}' = \mathcal{H}_2^\ell$. We will show that \mathcal{H}' satisfies **A1–A3**, and, with probability at least $1 - D^{-10}$, **A4** holds.

A1: Case 1. Let $e \in E(\mathcal{H})$ be such that $e \in B_e$ for some $B_e \in \mathcal{B}$ and there is some $v_e \in V$ and $B'_e \in \mathcal{B}_{v_e}$ with $e \in B'_e$. For each $h \in [\ell]$, we will show that $\mathbb{P}(e \in E(\mathcal{H}_1^h) | e \in E(\mathcal{H}_0^h)) = 1/8$. Suppose then that $h \in [\ell]$ and $e \in E(\mathcal{H}_0^h)$. Note that, as $e \in E(\mathcal{H}_0^h)$, from **C1** we have that U_{B_e} and $v_e B'_e$ are not in the same path/cycle $S_{i,j}^h$ for any $i \in [k/2^h]$ and $j \in [r_i^h]$. Furthermore, $e \in E(\mathcal{H}_1^h)$ only if $U_{B_e} \in M^h$, $v_e B'_e \in M^h$ and $e \in A_{B_e}^h$, which are three independent events which each occur with probability $1/2$. Thus, $\mathbb{P}(e \in E(\mathcal{H}_1^h) | e \in E(\mathcal{H}_0^h)) = 1/8$. Therefore, $\mathbb{P}(e \in E(\mathcal{H}')) \leq \mathbb{P}(e \in E(\mathcal{H}_1^\ell)) \leq 1/8^\ell = 1/k^3 = 1/t$.

Case 2. Let $e \in E(\mathcal{H})$ be such that for each $v \in V(e)$ there is some $B_{e,v} \in \mathcal{B}_v$ with $e \in B_{e,v}$. For each $h \in [\ell]$, we will show that $\mathbb{P}(e \in E(\mathcal{H}_1^h) | e \in E(\mathcal{H}_0^h)) = 1/8$. Suppose then that $h \in [\ell]$ and $e \in E(\mathcal{H}_0^h)$. Note that, as $e \in E(\mathcal{H}_0^h)$, from **C2** we have that no two of $v B_{e,v}$ are in the same path/cycle $S_{i,j}^h$ for any $i \in [k/2^h]$ and $j \in [r_i^h]$. Furthermore, $e \in E(\mathcal{H}_1^h)$ only if $v B_{e,v} \in M^h$ for each $v \in V(e)$, which are three independent events which each occur with probability $1/2$. Thus, $\mathbb{P}(e \in E(\mathcal{H}_1^h) | e \in E(\mathcal{H}_0^h)) = 1/8$. Therefore, $\mathbb{P}(e \in E(\mathcal{H}')) \leq \mathbb{P}(e \in E(\mathcal{H}_1^\ell)) \leq 1/8^\ell = 1/k^3 = 1/t$.

A2: Let $v \in V$. Note that if there is no edge in M that contains v then there are no edges in \mathcal{H}' containing v , and thus $d_{\mathcal{H}'}(v) = 0$. As there is at most one edge in M which contains v , we can assume that there is exactly one such edge. Suppose that this edge corresponds to U_B for some $B \in \mathcal{B}$. Then, the only edges $e \in E(\mathcal{H}')$ with $v \in V(e)$ are those in B , and therefore, by **D1**,

$$d_{\mathcal{H}'}(v) \leq |B \cap E(\mathcal{H}_2^\ell)| \leq (1 + D^{-\varepsilon})m/2^{2\ell} = (1 + D^{-\varepsilon})D/k^3 = (1 + D^{-\varepsilon})D/t.$$

Suppose then that the edge in M containing v is vB for some $B \in \mathcal{B}_v$. Then, similarly by **D1**, $d_{\mathcal{H}'}(v) \leq |B \cap E(\mathcal{H}_2^\ell)| \leq (1 + D^{-\varepsilon})D/t$. Thus, **A2** holds.

A3: Let $x, y \in V(\mathcal{H})$ be distinct. Suppose there is some $B \in \mathcal{B}$ with $U_B = \{x, y\}$ and $U_B \in M$. Let $e \in E(\mathcal{H}')$ contain x . Then, M must contain an edge which contains x which is either $U_{B'}$ for some $B' \in \mathcal{B}$ or $x B'$ for some $B' \in \mathcal{B}_x$, for which $e \in B'$. As M is a matching, the only possibility is $B' = B$, and then $\{x, y\} \subset V(e)$. Thus, there are no edges in \mathcal{H}' that contain x but not y . Arguing similarly, there are also no edges in \mathcal{H}' that contain y but not x . Thus, every edge in \mathcal{H}' contains either both x and y or neither of them.

Now, suppose there is no $B \in \mathcal{B}$ with $U_B = \{x, y\}$ and $U_B \in M$. If there is some $B \in \mathcal{B}$ with $x \in U_B$ and $U_B \in M$, then, if z is such that $U_B = \{x, z\}$, then the only possible edge in \mathcal{H}' containing x and y is xyz , and thus $\text{cod}_{\mathcal{H}'}(x, y) \leq 1$. Similarly, if there is some $B \in \mathcal{B}$ with $y \in U_B$ and $U_B \in M$, then $\text{cod}_{\mathcal{H}'}(x, y) \leq 1$. Therefore, if $\text{cod}_{\mathcal{H}'}(x, y) > 1$, there must be some $B \in \mathcal{B}_x$ and $B' \in \mathcal{B}_y$ with $x B, y B' \in M$, so that, from **D3**, $\text{cod}_{\mathcal{H}'}(x, y) \leq |(B \cap B') \cap E(\mathcal{H}_2^\ell)| \leq 2D^\gamma m/2^\ell k^2 = 2D^{\gamma-\mu} D/t \leq D^{1-\varepsilon}/t$, where we have used that $\varepsilon \ll \gamma \ll \mu$. This completes the proof of **A3**.

A4: It is left then only to prove that **A4** holds with probability at least $1 - D^{-10}$. We start by showing the following claim.

Claim 4.6. For each $h \in [\ell]$ and each distinct $v, w \in V$ and $B \in \mathcal{B}_v, B' \in \mathcal{B}_w$, with probability at least $1 - D^{-20}$,

$$|(B \cap B') \cap E(\mathcal{H}_1^h)| \leq (1 + h/\ell)D^\gamma m/2^h k^2. \quad (1)$$

Proof of Claim 4.6. Let $h \in [\ell]$. Let $v, w \in V$ be distinct and let $B \in \mathcal{B}_v$ and $B' \in \mathcal{B}_w$. Now, by the choice of \mathcal{H}_2^{h-1} (in particular **B3.3** or **D3**), we always have that

$$|(B \cap B') \cap E(\mathcal{H}_0^h)| \leq |(B \cap B') \cap E(\mathcal{H}_2^{h-1})| \leq (1 + (h-1)/2\ell)D^\gamma m/2^{h-1} k^2. \quad (2)$$

Let $Z_{v,w}$ be the set of z such that $vwz \in (B \cap B') \cap E(\mathcal{H}_0^h)$. For each $z \in Z_{v,w}$, let $B_z \in \mathcal{B}_z$ be such that $vwz \in B_z$, noting that, therefore, $zB_z \in M^{h-1}$. For each $z \in Z_{v,w}$, note that, if $vwz \in E(\mathcal{H}_1^h)$, then we must have that the indicator variable for the cycles/paths containing zB_z must be equal to 1 or 0 depending on which matching zB_z is in in $M^{h-1} = M_1^{h-1} \cup \dots \cup M_{k/2^{h-1}}^{h-1}$. Therefore, using (2),

$$\mathbb{E}|(B \cap B') \cap E(\mathcal{H}_1^h)| \leq (1 + (h-1)/2\ell)D^\gamma m/2^h k^2.$$

For each $i \in [k/2^h]$ and $j \in [r_i^h]$, let $c_{i,j}^h$ be the number of edges in $S_{i,j}^h$ which are zB_z for some $z \in Z_{v,w}$, and note that changing the variable $x_{i,j}^h$ changes $|(B \cap B') \cap E(\mathcal{H}_1^h)|$ by at most $c_{i,j}^h$, and that $c_{i,j}^h \leq s$. Thus,

$$\sum_{i \in [k/2^h]} \sum_{j \in [r_i^h]} (c_{i,j}^h)^2 \leq s \cdot \sum_{i \in [k/2^h]} \sum_{j \in [r_i^h]} c_{i,j}^h \leq s|Z_{v,w}| \stackrel{(2)}{\leq} 2sD^\gamma m/2^{h-1}k^2.$$

Therefore, by Lemma 4.4, (1) does not hold with probability at most

$$2 \exp\left(-\frac{(\frac{1}{\ell}D^\gamma m/2^h k^2)^2}{2sD^\gamma m/2^{h-1}k^2}\right) = 2 \exp\left(-\frac{D^\gamma m/2^h k^2}{4s\ell^2}\right) \leq 2 \exp\left(-\frac{D^{1+\gamma}}{4s\ell^2 k^4}\right) \leq D^{-20},$$

as required. \square

Now we show a similar claim to Claim 4.6 for codegrees.

Claim 4.7. For each $h \in [\ell]$ and each distinct $v, w \in V$ and $B \in \mathcal{B}_v$, with probability at least $1 - D^{-20}$, the number of edges in $B \cap E(\mathcal{H}_1^h)$ containing w is at most $(1 + h/\ell)D^{2\gamma}m/2^{2h}k$.

Proof of Claim 4.7. Let $h \in [\ell]$. Let $v, w \in V$ be distinct and let $B \in \mathcal{B}_v$. Now, by the choice of \mathcal{H}_2^{h-1} (in particular **B3.2** or **D2**), we always have that the number of edges in $B \cap E(\mathcal{H}_0^h)$ containing w is at most $(1 + (h-1)/\ell)D^{2\gamma}m/2^{2(h-1)}k$.

Let $Z_{v,w}$ be the set of $z \in V \setminus \{v, w\}$ such that $vwz \in B \cap E(\mathcal{H}_0^h)$ and there is some $B_z \in \mathcal{B}$ such that $vwz \in B_z$ and $z \in U_{B_z}$. Note that, as $vwz \in B$, then, by **B3.1**, $U_{B_z} = \{z, w\}$. Let $Z'_{v,w}$ be the set of $z \in V \setminus \{v, w\}$ such that $vwz \in B \cap E(\mathcal{H}_0^h)$ and there is some $B_z \in \mathcal{B}_z$ with $vwz \in B_z$ and some $B_z^w \in \mathcal{B}_w$ with $vwz \in B_z^w$. Note that $|Z_{v,w}| + |Z'_{v,w}| \leq (1 + (h-1)/\ell)D^{2\gamma}m/2^{2(h-1)}k$.

Now, for each $z \in Z_{v,w}$, we have $U_{B_z} \in M^{h-1}$ and $U_{B_z} \notin F^h$ by the choice of \mathcal{H}_0^h (from **C1**). Note that if $vwz \in E(\mathcal{H}_1^h)$ then we have $vwz \in A_{B_z}^h$ and the indicator variable $x_{i,j}^h$ for the path/cycle containing U_{B_z} must be equal to 0 or 1 depending on which matching U_{B_z} is in in the partition $M^{h-1} = M_1^{h-1} \cup \dots \cup M_{k/2^{h-1}}^{h-1}$. Thus, $vwz \in E(\mathcal{H}_1^h)$ with probability 1/4.

Furthermore, for each $z \in Z'_{v,w}$, from **C2** we have $zB_z, wB_z^w \in M^{h-1} \setminus F^h$, and that zB_z and wB_z^w do not appear in the same path/cycle in $S_{i,j}^h$, $i \in [k/2^h]$ and $j \in [r_i^h]$. Note that if $vwz \in E(\mathcal{H}_1^h)$ then the indicator variables $x_{i,j}^h$ for the path/cycles containing zB_z and wB_z^w must be equal to 0 or 1 depending on which matching zB_z and wB_z^w is in respectively in the partition $M^{h-1} = M_1^{h-1} \cup \dots \cup M_{k/2^{h-1}}^{h-1}$. Thus, $vwz \in E(\mathcal{H}_1^h)$ with probability 1/4.

Therefore, altogether, we have

$$\mathbb{E}\{|e \in B \cap E(\mathcal{H}_1^h) : w \in V(e)\}| = \frac{1}{4}(|Z_{v,w}| + |Z'_{v,w}|) \leq (1 + (h-1)/2\ell)D^{2\gamma}m/2^{2h}k.$$

Now, for each $i \in [k/2^h]$ and $j \in [r_i^h]$, let $c_{i,j}^h$ be the number of $z \in Z_{v,w}$ with U_{B_z} in the path/cycle $S_{i,j}^h$ or $z \in Z'_{v,w}$ with zB_z or wB_z^w in the path/cycle $S_{i,j}^h$. Then, changing the variable $x_{i,j}^h$ changes $\{|e \in B \cap E(\mathcal{H}_1^h) : w \in V(e)\}|$ by at most $c_{i,j}^h$. Note that we have $c_{i,j}^h \leq s \cdot 2D^\gamma m/2^{h-1}k^2$ by the choice of \mathcal{H}_2^{h-1} (in particular **B3.3** or **D3**). Thus,

$$\begin{aligned} \sum_{i \in [k/2^h]} \sum_{j \in [r_i^h]} (c_{i,j}^h)^2 &\leq (s \cdot 2D^\gamma m/2^{h-1}k^2) \cdot \sum_{i \in [k/2^h]} \sum_{j \in [r_i^h]} c_{i,j}^h \\ &\leq (s \cdot 2D^\gamma m/2^{h-1}k^2) \cdot (|Z_{v,w}| + 2|Z'_{v,w}|) \\ &\leq (s \cdot 2D^\gamma m/2^{h-1}k^2) \cdot 4D^{2\gamma}m/2^{2(h-1)}k = 8sD^{3\gamma}m^2/2^{3(h-1)}k^3. \end{aligned}$$

Furthermore, for each $z \in Z_{v,w}$, changing whether or not $vwz \in A_{B_z}^h$ changes $\{e \in B \cap E(\mathcal{H}_1^h) : w \in V(e)\}$ by at most 1, and $|Z_{v,w}| + \sum_{i \in [k/2^h]} \sum_{j \in [r_i^h]} (c_{i,j}^h)^2 \leq 16sD^{3\gamma}m^2/2^{3(h-1)}k^3$. Thus, by Lemma 4.4, the number of edges in $B \cap E(\mathcal{H}_1^h)$ containing w is more than $(1 + h/\ell)D^{2\gamma}m/2^{2h}k$ with probability at most

$$2 \exp \left(- \frac{(\frac{1}{\ell}D^{2\gamma}m/2^{2h}k)^2}{16sD^{3\gamma}m^2/2^{3(h-1)}k^3} \right) = 2 \exp \left(- \frac{D^\gamma k}{128s\ell^2 2^h} \right) \leq 2 \exp \left(- \frac{D^{\gamma/2}}{128\ell^2} \right) \leq D^{-20},$$

where we have used that $s = D^{\gamma/2}$ and $2^h \leq 2^\ell = k$. \square

Now we show a similar claim to Claim 4.7 for degrees.

Claim 4.8. For each $B \in \mathcal{B} \cup (\cup_{v \in V} \mathcal{B}_v)$, with probability at least $1 - D^{-20}$,

$$|B \cap E(\mathcal{H}_1^h)| \leq (1 + D^{-\varepsilon}h/\ell)m/2^{2h}.$$

Proof of Claim 4.8. Case 1. Let $B \in \mathcal{B}$. Now, by the choice of \mathcal{H}_2^{h-1} (in particular **B3.1** or **D1**), we always have that $|B \cap E(\mathcal{H}_0^h)| \leq (1 + D^{-\varepsilon}(h-1)/\ell)m/2^{2(h-1)}$. Let Z_B be the set of z such that $\{z\} \cup U_B \in B \cap E(\mathcal{H}_0^h)$, so that $|Z_B| \leq (1 + D^{-\varepsilon}(h-1)/\ell)m/2^{2(h-1)}$. For each $z \in Z_B$, let $B_z \in \mathcal{B}_z$ be such that $\{z\} \cup U_B \in B_z$, noting that, therefore, $zB_z \in M^{h-1} \setminus F_h$. Note that if $\{z\} \cup U_B \in B \cap E(\mathcal{H}_1^h)$, then we must have that the indicator variable for the cycle/path containing zB_z must be equal to 1 or 0 depending on which matching zB_z is in in the partition $M^{h-1} = M_1^{h-1} \cup \dots \cup M_{k/2^{h-1}}^{h-1}$, and we must have $\{z\} \cup U_B \in A_B^h$. Therefore,

$$\mathbb{E}|B \cap E(\mathcal{H}_1^h)| = \frac{1}{4}|Z_B| \leq (1 + D^{-\varepsilon}(h-1)/2\ell)m/2^{2h}.$$

Now, changing whether or not a single edge is in A_B^h or not affects $|B \cap E(\mathcal{H}_1^h)|$ by at most 1, and at most $|Z_B| \leq 2m/2^{2(h-1)} = 8m/2^{2h}$ of these events can affect $|B \cap E(\mathcal{H}_1^h)|$. For each $i \in [k/2^h]$ and $j \in [r_i^h]$, let $c_{i,j}^h$ be the number of $z \in Z_B$ with zB_z in the path/cycle $S_{i,j}^h$. Then, changing the variable $x_{i,j}^h$ changes $|B \cap E(\mathcal{H}_1^h)|$ by at most $c_{i,j}^h \leq s$, while $\sum_{i \in [k/2^h]} \sum_{j \in [r_i^h]} c_{i,j}^h \leq |Z_B|$. Thus

$$\sum_{i \in [k/2^h]} \sum_{j \in [r_i^h]} (c_{i,j}^h)^2 \leq s \cdot \sum_{i \in [k/2^h]} \sum_{j \in [r_i^h]} c_{i,j}^h \leq s \cdot |Z_B| \leq s \cdot 2m/2^{2(h-1)} = 8sm/2^{2h}.$$

Thus, by Lemma 4.4, $|B \cap E(\mathcal{H}_1^{h-1})| > (1 + D^{-\varepsilon}h/\ell)m/2^{2h}$ with probability at most

$$\begin{aligned} 2 \exp \left(- \frac{(\frac{1}{\ell}D^{-\varepsilon}m/2^{2h})^2}{8sm/2^{2h} + 8m/2^{2h}} \right) &\leq 2 \exp \left(- \frac{(\frac{1}{\ell}D^{-\varepsilon}m/2^{2h})^2}{16sm/2^{2h}} \right) = 2 \exp \left(- \frac{D^{-2\varepsilon}m}{2^{2h} \cdot \ell^2 \cdot 16s} \right) \\ &\leq 2 \exp \left(- \frac{D^{1-2\varepsilon}}{16k^3\ell^2s} \right) \leq D^{-20}, \end{aligned}$$

as required.

Case 2. Let then $B \in \mathcal{B}_v$ with $v \in V$. Now, by the choice of \mathcal{H}_2^{h-1} (in particular **B3.1** or **D1**), we always have that $|B \cap E(\mathcal{H}_0^h)| \leq (1 + D^{-\varepsilon}(h-1)/\ell)m/2^{2(h-1)}$. Let $E_B = (B \cap E(\mathcal{H}_0^h)) \cap (\cup_{B' \in \mathcal{B}} B')$, and let $E'_B = (B \cap E(\mathcal{H}_0^h)) \setminus E_B$. Note that $|E_B| + |E'_B| \leq (1 + D^{-\varepsilon}(h-1)/\ell)m/2^{2(h-1)}$. For each $e \in E_B$, let $B_e \in \mathcal{B}$ be such that $e \in B_e$, and note that $U_{B_e} \in M^{h-1}$. For each $e \in E'_B$, let w_e, z_e be such that $e = vw_ez_e$, and let $B_e^w \in \mathcal{B}_{w_e}$ and $B_e^z \in \mathcal{B}_{z_e}$ satisfy $e \in B_e^w$ and $e \in B_e^z$, and note that $w_eB_e^w$ and $z_eB_e^z$ are in M^{h-1} but in different cycles/paths $S_{i,j}^h$ across $i \in [k/2^h]$ and $j \in [r_i^h]$.

For each $e \in E_B$, note that if $e \in B \cap E(\mathcal{H}_1^h)$ then we must have that the indicator variable for the cycle/path containing B_e must be equal to 1 or 0 depending on which matching U_{B_e} is in in the partition $M^{h-1} = M_1^{h-1} \cup \dots \cup M_{k/2^{h-1}}^{h-1}$ and that $e \in A_{B_e}^h$, which happens with probability 1/4. For each $e \in E'_B$ note that if $e \in B \cap E(\mathcal{H}_1^h)$ then we must have that the indicator variables for the cycles/paths containing

$w_e B_{w_e}$ and $z_e B_{z_e}$ must be 0 or 1 depending on which matching $w_e B_{w_e}$ and $z_e B_{z_e}$ are in in the partition $M^{h-1} = M_1^{h-1} \cup \dots \cup M_{k/2^{h-1}}^{h-1}$, which again happens with probability $1/4$. Therefore,

$$\mathbb{E}|B \cap E(\mathcal{H}_1^h)| = \frac{1}{4}(|E_B| + |E'_B|) \leq (1 + D^{-\varepsilon}(h-1)/2\ell)m/2^{2h}.$$

Now, changing whether or not a single edge e is in $A_{B_e}^h$ or not affects $|B \cap E(\mathcal{H}_1^h)|$ by at most 1 and at most $|E_B| \leq 2m/2^{2(h-1)}$ such events affect $|B \cap E(\mathcal{H}_1^h)|$. For each $i \in [k/2^h]$ and $j \in [r_i^h]$, let $c_{i,j}^h$ be the number of $e \in E_B$ with U_{B_e} in the path/cycle $S_{i,j}^h$ or $e \in E'_B$ with $w_e B_{w_e}$ or $z_e B_{z_e}$ in the path/cycle $S_{i,j}^h$. Now, for each $w B' \in E(S_{i,j}^h)$ with $w \in V \setminus \{v\}$ and $B' \in \mathcal{B}_w$, there are, by **D2**, at most $2D^{2\gamma}m/2^{2(h-1)}k$ edges $e \in B \cap E(\mathcal{H}_0^h)$ with $w_e = w$ and $B_{w_e} = B'$. Furthermore, for each distinct $e, e' \in E_B \cap E(S)$ we have $B_e \neq B_{e'}$ as $e = \{v\} \cup U_{B_e}$ and $e' = \{v\} \cup U_{B_{e'}}$. Therefore, as $e(S_{i,j}^h) \leq s$, we have $c_{i,j}^h \leq 8sD^{2\gamma}m/2^{2h}k$. Changing the variable $x_{i,j}^h$ changes $|B \cap E(\mathcal{H}_1^h)|$ by at most $c_{i,j}^h$, while

$$\begin{aligned} \sum_{i \in [k/2^h]} \sum_{j \in [r_i^h]} (c_{i,j}^h)^2 &\leq (8sD^{2\gamma}m/2^{2h}k) \cdot \sum_{i \in [k/2^h]} \sum_{j \in [r_i^h]} c_{i,j}^h \leq (8sD^{2\gamma}m/2^{2h}k) \cdot (|E_B| + 2|E_{B'}|) \\ &\leq (8sD^{2\gamma}m/2^{2h}k) \cdot 4m/2^{2(h-1)} = 2^7 s D^{2\gamma} m^2 / 2^{4h} k, \end{aligned}$$

so $|E_B| + \sum_{i \in [k/2^h]} \sum_{j \in [r_i^h]} (c_{i,j}^h)^2 \leq 2m/2^{2(h-1)} + 2^7 s D^{2\gamma} m^2 / 2^{4h} k \leq 2^8 s D^{2\gamma} m^2 / 2^{4h} k$. Thus, by Lemma 4.4, $|B \cap E(\mathcal{H}_1^h)| < (1 + D^{-\varepsilon}h/2\ell)m/2^{2h}$ with probability at most

$$2 \exp\left(-\frac{(\frac{1}{\ell}D^{-\varepsilon}m/2^{2h})^2}{2^8 s D^{2\gamma} m^2 / 2^{4h} k}\right) = 2 \exp\left(-\frac{D^{-2\varepsilon-2\gamma}k}{2^8 s \ell^2}\right) \leq D^{-20},$$

where we have used that $k = D^\mu$ and $\varepsilon \ll \gamma \ll \mu$. \square

Putting Claims 4.6 to 4.8 together allows us to show that there is likely to be only an edge loss from \mathcal{H}_1^h to \mathcal{H}_2^h , as follows.

Claim 4.9. For each $h \in [\ell]$, with probability at least $1 - D^{-11}$,

$$e(\mathcal{H}_2^h) \geq e(\mathcal{H}_1^h) - |V|. \quad (3)$$

Proof of Claim 4.9. Let $h \in [\ell]$. Let X^h be the set of vertices $v \in V$ for which at least one of the following does not hold.

- There is some $B \in \mathcal{B}_v \cup \{B' \in \mathcal{B} : v \in U_{B'}\}$ such that $|B \cap E(\mathcal{H}_2^h)| \leq (1 + h \cdot D^{-\varepsilon}/\ell)m/2^{2h}$.
- There is some $w \in V \setminus \{v\}$ and $B \in \mathcal{B}_w$, such that the number of edges in $B \cap E(\mathcal{H}_2^h)$ containing w is at most $(1 + h/\ell)D^{2\gamma}m/2^{2h}k$.
- There is some $w \in V \setminus \{v\}$, $B \in \mathcal{B}_w$, and $B' \in \mathcal{B}_w$ with $|(B \cap B') \cap E(\mathcal{H}_2^h)| \leq (1 + h/\ell)D^\gamma m/2^h k^2$.

Note that, as $\Delta(\mathcal{H}) \leq D$, using Claims 4.6 to 4.8, we have that, for each $v \in V$, $\mathbb{P}(v \in X^h) \leq 3D^2 \cdot D^{-20}$. Let E^h be the set of edges of \mathcal{H} containing some vertex in X^h , so that $\mathbb{E}|E^h| \leq |V| \cdot D \cdot 3D^2 \cdot D^{-20}$. Then, by Markov's inequality, we have $|E^h| \geq |V|$ with probability at most D^{-11} . Finally, note that if $\mathcal{H}_1^h - E^h$ replaces \mathcal{H}_2^h then **D1–D3** are satisfied. Thus, by the maximality of \mathcal{H}_2^h , $e(\mathcal{H}_2^h) \geq e(\mathcal{H}_1^h) - |E^h|$, and therefore (3) holds with probability at least $1 - D^{-11}$. \square

We now show that there will always be only a small edge loss from \mathcal{H}_2^{h-1} to \mathcal{H}_0^h .

Claim 4.10. For each $h \in [\ell]$,

$$e(\mathcal{H}_0^h) \geq e(\mathcal{H}_2^{h-1}) - |V|D^{1-\gamma/3}/2^{3h}$$

Proof of Claim 4.10. Let $h \in [\ell]$. Note that by the minimality in the choice of F^h we have

$$|F^h| \leq \frac{2|V|}{s} \cdot \frac{k}{2^h}.$$

Furthermore, for each $B \in \mathcal{B}$ with $U_B \in F^h$, $|B \cap E(\mathcal{H}_2^{h-1})| \leq 2m/2^{2(h-1)}$ by the choice of \mathcal{H}_2^{h-1} (in particular **B2** or **D1**). Similarly, for each $v \in V$ and $B \in \mathcal{B}_v$ with $vB \in F^h$, we have that $|B \cap E(\mathcal{H}_2^{h-1})| \leq 2m/2^{2(h-1)}$ by the choice of \mathcal{H}_2^{h-1} . Therefore, the number of edges $e \in E(\mathcal{H}_2^{h-1})$ for which either $e \in \cup_{B \in \mathcal{B}: U_B \in F^h} B$ or $e \in B$ for some $v \in V$, $B \in \mathcal{B}_v$ and $vB \in F^h$ is at most

$$|F^h| \cdot 2m/2^{2(h-1)} \leq \frac{2|V|}{s} \cdot \frac{k}{2^h} \cdot \frac{8m}{2^{2h}} \leq \frac{16|V|D}{s2^{3h}}. \quad (4)$$

Let $i \in [k/2^h]$ and $j \in [r_i^h]$. For each $B \in \mathcal{B}$, $v \in V$ and $B' \in \mathcal{B}_v$ with $U_B, vB' \in S_{i,j}^h$, we have $B \cap B' = \emptyset$ if $v \in U_B$ and otherwise the only possible edge in $B \cap B'$ is $\{v\} \cup U_B$, so in either case $|B \cap B'| \leq 1$. For each distinct $v, w \in V$ and $B \in \mathcal{B}_v, B' \in \mathcal{B}_w$, we have $|(B \cap B') \cap E(\mathcal{H}_2^{h-1})| \leq 2D^\gamma m/2^{h-1}k^2$ by the choice of \mathcal{H}_2^{h-1} (in particular **B3.3** or **D3**).

Therefore, using the maximality of \mathcal{H}_h^0 , we have

$$\begin{aligned} e(\mathcal{H}_2^{h-1}) - e(\mathcal{H}_0^h) &\leq \frac{16|V|D}{s2^{3h}} + \sum_{i \in [k/2^h]} \sum_{j \in [r_i^h]} \frac{e(S_{i,j}^h)^2 \cdot 2D^\gamma m}{2^{h-1}k^2} \leq \frac{16|V|D}{s2^{3h}} + \sum_{i \in [k/2^h]} \sum_{j \in [r_i^h]} s \cdot \frac{2e(S_{i,j}^h) \cdot 2D^{1+\gamma}}{2^h k^3} \\ &\leq \frac{16|V|D}{s2^{3h}} + \sum_{i \in [k/2^h]} \frac{8s \cdot |V| \cdot D^{1+\gamma}}{2^h k^3} = \frac{16|V|D}{s2^{3h}} + \frac{8s \cdot |V| \cdot D^{1+\gamma}}{2^{2h} \cdot k^2} \\ &= \frac{|V|D}{2^{3h}} \cdot \left(\frac{16}{s} + \frac{8sD^\gamma 2^h}{k^2} \right) \leq \frac{|V|D}{2^{3h}} \cdot \left(\frac{16}{s} + \frac{8sD^\gamma}{k} \right) \leq D^{-\gamma/3} \cdot \frac{|V|D}{2^{3h}}, \end{aligned}$$

as required, where we have used that $k = D^\mu$, $s = D^{\gamma/2}$, and $\gamma \ll \mu$. \square

Now we show that it is likely that the edge loss from \mathcal{H}_0^h to \mathcal{H}_1^h is only at most slightly larger than the expected factor of $7/8$.

Claim 4.11. For each $h \in [\ell]$, with probability at least $1 - D^{-11}$,

$$e(\mathcal{H}_1^h) \geq \frac{1}{8}e(\mathcal{H}_0^h) - D^{1-2\varepsilon}|V|/2^{3h}. \quad (5)$$

Proof of Claim 4.11. Let E be the set of edges $e \in E(\mathcal{H}_0^h)$ for which $e \in \cup_{B \in \mathcal{B}} B$, and let $E' = E(\mathcal{H}_0^h) \setminus E$. For each $e \in E$, let $B_e \in \mathcal{B}$, $z_e \in V \setminus U_{B_e}$ and $B'_e \in \mathcal{B}_{z_e}$ be such that $e \in B_e \cap B'_e$. Note that $e \in E(\mathcal{H}_1^h)$ exactly if $U_{B_e}, z_e B' \in M^h$ and $e \in A_{B_e}^h$, which, by the choice of \mathcal{H}_0^h , happens with probability $1/8$. For each $e \in E'$, label vertices v_e, w_e, z_e and take $B_e^v \in \mathcal{B}_{v_e}, B_e^w \in \mathcal{B}_{w_e}$ and $B_e^z \in \mathcal{B}_{z_e}$ such that $e = \{v_e, w_e, z_e\}$ and $e \in B_e^v \cap B_e^w \cap B_e^z$. Note that $e \in E(\mathcal{H}_1^h)$ exactly if $v_e B_e^v, w_e B_e^w, z_e B_e^z \in M^h$, which, by the choice of \mathcal{H}_0^h , happens with probability $1/8$. Therefore, $\mathbb{E}(e(\mathcal{H}_1^h)) \geq \frac{1}{8}e(\mathcal{H}_0^h)$.

Now, changing whether or not a single edge is in A_B^h or not affects $e(\mathcal{H}_1^h)$ by at most 1, and the number of these variables is (using the choice of \mathcal{H}_0^h , and in particular **B3.1** or **D1**) at most

$$e(\mathcal{H}_0^h) \leq |V| \cdot (k/2^{h-1}) \cdot 2m/2^{2(h-1)} = 16|V|D/2^{3h}.$$

For each $i \in [k/2^h]$ and $j \in [r_i^h]$, changing the variable $x_{i,j}^h$ changes $e(\mathcal{H}_1^h)$ by (again using the choice of \mathcal{H}_0^h , and in particular **B3.1** or **D1**) at most $c_{i,j}^h := e(S_{i,j}^h) \cdot 2m/2^{2(h-1)}$. Note that

$$\sum_{i \in [k/2^h]} \sum_{j \in [r_i^h]} (c_{i,j}^h)^2 \leq \sum_{i \in [k/2^h]} \sum_{j \in [r_i^h]} e(S_{i,j}^h) \cdot s \left(\frac{2m}{2^{2(h-1)}} \right)^2 \leq \frac{k}{2^h} \cdot 2|V| \cdot s \left(\frac{2m}{2^{2(h-1)}} \right)^2 = \frac{2^7 D^2 s |V|}{2^{5h} k},$$

so that

$$e(\mathcal{H}_0^h) + \sum_{i \in [k/2^h]} \sum_{j \in [r_i^h]} (c_{i,j}^h)^2 \leq \frac{16|V|D}{2^{3h}} + \frac{2^7 D^2 s |V|}{2^{5h} k} \leq \frac{2^8 D^2 s |V|}{2^{5h} k}.$$

Thus, by Lemma 4.4, (5) does not hold with probability at most

$$2 \exp\left(-\frac{(D^{1-2\varepsilon}|V|/2^{3h})^2}{2^8 D^2 s |V|/2^{5h} k}\right) = 2 \exp\left(-\frac{D^{-4\varepsilon} k |V|}{2^8 2^h s}\right) \leq 2 \exp\left(-\frac{D^{-4\varepsilon-\gamma/2} |V|}{2^8}\right) \leq D^{-11},$$

where we have used that $|V| \geq D^{1/2}$. \square

Claims 4.9–4.11 allows us to bound the likely edge loss from \mathcal{H}_1 to \mathcal{H}' . Indeed, using Claims 4.9–4.11, we have, with probability at least $1 - 3\ell D^{-11} \geq 1 - D^{-10}$, that, for each $h \in [\ell]$, $e(\mathcal{H}_0^h) \geq e(\mathcal{H}_2^{h-1}) - D^{1-\gamma/3} \cdot |V|/2^{3h}$, $e(\mathcal{H}_1^h) \geq \frac{1}{8}e(\mathcal{H}_0^h) - D^{1-2\varepsilon}|V|/2^{3h}$ and $e(\mathcal{H}_2^h) \geq e(\mathcal{H}_1^h) - |V|$. When this holds, we have, for each $h \in [\ell]$, that

$$e(\mathcal{H}_2^h) \geq \frac{1}{8}e(\mathcal{H}_2^{h-1}) - 3D^{1-2\varepsilon}|V|/2^{3h},$$

and therefore, it is easy to show by induction that, for each $h \in [\ell]$, $e(\mathcal{H}_2^h) \geq 2^{-3h} \cdot e(\mathcal{H}_0^{h-1}) - h \cdot 3D^{1-2\varepsilon}|V|/2^{3h}$. Thus, we have

$$e(\mathcal{H}') = e(\mathcal{H}_2^\ell) \geq 2^{-3\ell} \cdot e(\mathcal{H}_2^0) - \ell \cdot 3D^{1-2\varepsilon}|V|/2^{3\ell} \geq \frac{1}{t}e(\mathcal{H}_2^0) - \frac{1}{2t}D^{1-\varepsilon}|V|.$$

To complete the proof of **A4**, it is sufficient then to show that $e(\mathcal{H}_1) = e(\mathcal{H}_2^0) \geq e(\mathcal{H}) - D^{1-\varepsilon}|V|/2$.

For this, first note that, from **B1**, for each $v \in V$, there are at most $D/m \leq 2D^\mu$ vertices $x \in V$ with $\text{cod}_{\mathcal{H}}(v, x) \geq m$, and, thus, $e(\mathcal{H}) - e(\mathcal{H}_0) \leq |V| \cdot (2D^\mu)^2$. Take an arbitrary \mathcal{B} satisfying **B2**. For each $v \in V$, let

$$k_v = \lfloor |\{e \in E(\mathcal{H}_0) \setminus \cup_{B \in \mathcal{B}: v \in U_B} B : v \in V(e)\}|/m \rfloor$$

and, if $k_v \geq k \cdot D^{-\gamma/4}$, then uniformly at random take a collection \mathcal{B}_v of k_v disjoint subsets of size m in $\{e \in E(\mathcal{H}_0) \setminus \cup_{B \in \mathcal{B}: v \in U_B} B : v \in V(e)\}$, and, otherwise, take $\mathcal{B}_v = \emptyset$.

Now, letting $Z \subseteq E(\mathcal{H}_0)$ to be the set of edges $e \in E(\mathcal{H}_0)$ for which either there is some $B \in \mathcal{B}$ with $e \in B$ and some $v \in V$ and $B' \in \mathcal{B}_v$ with $e \in B'$ or, for each $v \in V(e)$, there is some $B \in \mathcal{B}_v$ with $e \in B$, we have

$$|E(\mathcal{H}_0) \setminus Z| \leq m \cdot (k \cdot D^{-\gamma/4} + 1) \cdot |V| \leq D^{1-\varepsilon}|V|/12.$$

Furthermore, for each distinct $v, w \in V$ with $k_v \geq k \cdot D^{-\gamma/4}$, as there are at most m edges in $E(\mathcal{H}_0) \setminus \cup_{B \in \mathcal{B}: v \in U_B} B$ containing both v and w by the maximality of \mathcal{B} , the probability that there is some $B \in \mathcal{B}_v$ for which there are at least $2m/k_v \leq D^{2\gamma}m/k$ edges in B containing w is, using an appropriate Chernoff bound for hypergeometric random variables, at most $k \cdot D^{-10}$. Similarly, for each distinct $v, w \in V$ with $k_v, k_w \geq k \cdot D^{-\gamma/4}$, the probability that there is some $B \in \mathcal{B}_v$ and $B' \in \mathcal{B}_w$ for which $|B \cap B'| \leq 2m/k_v k_w \leq D^{\gamma n}/k^2$ is at most $k^2 \cdot D^{-10}$.

Therefore, with positive probability, if we delete, for each $v \in V$, each $B \in \mathcal{B}_v$ for which **B3.2** or **B3.3** does not hold, then this deletes certainly at most $D^{1-\varepsilon}|V|/12$ edges. Thus, by the maximality of the \mathcal{B}_v , $v \in V$, we have that Z contains all but at most $D^{1-\varepsilon}|V|/6$ edges of $e(\mathcal{H}_0)$. Note that, by the definition of \mathcal{H}_1 , we therefore have that \mathcal{H}_1 contains all but at most $D^{1-\varepsilon}|V|/2$ of the edges of $e(\mathcal{H}_0)$. Thus, we have that $e(\mathcal{H}_1) \geq e(\mathcal{H}) - (2D^\mu)^2|V| - D^{1-\varepsilon}|V|/2 \geq e(\mathcal{H}) - D^{1-\varepsilon}|V|$, as required. \square

4.2 Proof of Theorem 4.3

We now use Lemma 4.5 and Corollary 4.2 to prove Theorem 4.3.

Proof of Theorem 4.3. Let $\varepsilon, \mu, \eta > 0$ and D_0 have the property from Lemma 4.5, as shown to exist by that lemma, let $\xi \ll \varepsilon, 1/k$, and, moreover, assume that $1/D_0 \ll \varepsilon, 1/k$. We will show that Theorem 4.3 holds for ξ, η and D_0 . Let then $D \geq D_0$ and let \mathcal{H} be a 3-uniform hypergraph with maximum degree D and at least \sqrt{D} vertices for which the graph on $V(\mathcal{H})$ with edges xy present if $\text{cod}_{\mathcal{H}}(x, y) \geq D^{1-\eta}$ is bipartite.

Using the property from Lemma 4.5, there is some $D^{3\mu} \leq t \leq 8D^{3\mu}$ for which, setting $r = t^2$, we can find independent random subhypergraphs $\mathcal{H}'_1, \dots, \mathcal{H}'_r$ of \mathcal{H} such that, for each $i \in [r]$,

- for each $e \in E(\mathcal{H})$, $\mathbb{P}(e \in E(\mathcal{H}'_i)) \leq 1/t$,

- $\Delta(\mathcal{H}'_i) \leq (1 + D^{-\varepsilon})D/t$,
- for each distinct $x, y \in V(\mathcal{H}'_i)$, either $\text{cod}_{\mathcal{H}'_i}(x, y) \leq D^{1-\varepsilon}/t$ or every $e \in E(\mathcal{H}')$ satisfies $\{x, y\} \subset V(e)$ or $\{x, y\} \cap V(e) = \emptyset$, and,
- with probability at least $1 - D^{-10}$, $e(\mathcal{H}'_i) \geq e(\mathcal{H})/t - D^{1-\varepsilon}|\mathcal{H}|/t$.

By a simple application of a Chernoff bound, we have, for each $e \in E(\mathcal{H})$, that

$$\mathbb{P}\left(|\{i \in [r] : e \in E(\mathcal{H}'_i)\}| > (1 + D^{-\varepsilon})\frac{r}{t}\right) < D^{-10}.$$

Therefore, by an application of Markov's lemma, with probability $1 - D^{-5}$, for all but at most $|\mathcal{H}|$ edges $e \in E(\mathcal{H})$ we have $|\{i \in [r] : e \in E(\mathcal{H}'_i)\}| \leq (1 + D^{-\varepsilon})\frac{r}{t}$. Let \mathcal{H}_1 be the subhypergraph of \mathcal{H} of these edges. With positive probability then, we can assume that $e(\mathcal{H}_1) \geq e(\mathcal{H}) - |\mathcal{H}|$, and that $e(\mathcal{H}'_i) \geq e(\mathcal{H})/t - D^{1-\varepsilon}|\mathcal{H}|/t$ for each $i \in [r]$.

Let $m = (1 + D^{-\varepsilon})r/t$. For each $e \in E(\mathcal{H}_1)$, independently at random pick $i_e \in \{0\} \cup \{i \in [r] : e \in E(\mathcal{H}'_i)\}$ so that $\mathbb{P}(i_e = i) = 1/m$ for each $i \in [r]$ for which $e \in E(\mathcal{H}'_i)$. For each $i \in [r]$, let \mathcal{H}''_i be the subhypergraph of \mathcal{H}'_i of edges $e \in E(\mathcal{H}'_i)$ with $i_e = i$. Using a Chernoff bound and the local lemma, we can assume that, for each $i \in [r]$, the following hold

- $\Delta(\mathcal{H}''_i) \leq (1 + D^{-\varepsilon/2})D/tm$,
- for each distinct $x, y \in V(\mathcal{H}''_i)$, either $\text{cod}_{\mathcal{H}''_i}(x, y) \leq D^{1-\varepsilon/2}/tm$ or every $e \in E(\mathcal{H}''_i)$ satisfies $\{x, y\} \subset V(e)$ or $\{x, y\} \cap V(e) = \emptyset$, and,
- $e(\mathcal{H}''_i) \geq e(\mathcal{H})/tm - 2D^{1-\varepsilon}|\mathcal{H}|/tm$.

Now, as $\xi, 1/D_0 \ll \varepsilon, 1/k$, by Corollary 4.2, $\chi(\mathcal{H}''_i) \leq (1 + D^{-\xi})D/tm$ for each $i \in [r]$. Let $\mathcal{H}' = \cup_{i \in [r]} \mathcal{H}''_i$, so that

$$\chi(\mathcal{H}') \leq r \cdot (1 + D^{-\xi})D/tm \leq (1 + D^{-\xi})D,$$

and

$$e(\mathcal{H}') \geq r \cdot e(\mathcal{H})/tm - r \cdot 2D^{1-\varepsilon}|\mathcal{H}|/tm = (1 + D^{-\varepsilon})^{-1} \cdot e(\mathcal{H}) - (1 + D^{-\varepsilon})^{-1} \cdot 2D^{1-\varepsilon}|\mathcal{H}| \geq e(\mathcal{H}) - |\mathcal{H}| \cdot D^{1-\varepsilon},$$

and hence \mathcal{H}' satisfies our requirements. \square

4.3 Proof of Theorem 1.1

We now deduce Theorem 1.1 from Theorem 4.3.

Proof of Theorem 1.1. Let $\xi, \eta > 0$ and D_0 be such that the property in Theorem 4.3 holds for each $D \geq D_0$ with ξ and η , and let $0 < \varepsilon \leq \xi/3$. Assume moreover that $\varepsilon > 0$ is small enough that $n - n^{1-\varepsilon} \leq 1$ for each $n < D_0$. We will show that Theorem 1.1 holds for ε . Let S then be an equi- n -square. Note that if $n < D_0$, then all that is required is for S to contain 1 transversal with at least 1 cell, which trivially holds. Assume then that $n \geq D_0$. By making ε even smaller if necessary, we may also assume $n^{\xi/3} > 2$.

Say the set of rows of S is \mathcal{I} , the set of columns is \mathcal{J} and the set of symbols is \mathcal{A} . Form a 3-uniform hypergraph \mathcal{H} on $\mathcal{I} \cup \mathcal{J} \cup \mathcal{A}$ by adding, for each $i \in \mathcal{I}$ and $j \in \mathcal{J}$, an edge ijc where $c \in \mathcal{A}$ is the symbol in the square of S indexed by (i, j) . If \mathcal{H} contains at least $n - n^{1-\varepsilon}$ disjoint matchings with at least $n - n^{1-\varepsilon}$ edges, then it can be easily seen that S has at least $n - n^{1-\varepsilon}$ disjoint transversals with size at least $n - n^{1-\varepsilon}$. Suppose then that \mathcal{H} does not contain at least $n - n^{1-\varepsilon}$ disjoint matchings with at least $n - n^{1-\varepsilon}$ edges.

Note that if $i \in \mathcal{I}$ and $j \in \mathcal{J}$ then $\text{cod}_{\mathcal{H}}(i, j) = 1$. Therefore, the graph on $V(\mathcal{H})$ with edges xy present if $\text{cod}_{\mathcal{H}}(x, y) \geq n^{1-\eta}$ is bipartite. Thus, by Theorem 4.3, there is some subhypergraph $\mathcal{H}' \subset \mathcal{H}$ with $\chi'(\mathcal{H}') \leq (1 + n^{-\xi})n$ and $e(\mathcal{H}') \geq e(\mathcal{H}) - n^{2-\xi}$. However, we then have

$$\begin{aligned} e(\mathcal{H}') &\leq (n - n^{1-\varepsilon})n + (\chi'(\mathcal{H}') - n + n^{1-\varepsilon})(n - n^{1-\varepsilon}) \\ &\leq (n - n^{1-\varepsilon})n + (n^{1-\xi} + n^{1-\varepsilon})(n - n^{1-\varepsilon}) \leq n^2 + n^{2-\xi} - n^{2-2\varepsilon}, \end{aligned}$$

so that $e(\mathcal{H}) \leq e(\mathcal{H}') + n^{2-\xi} \leq n^2 + 2n^{2-\xi} - n^{2-2\varepsilon} < n^2$, as $\varepsilon \leq \xi/3$ and we assumed $n^{\xi/3} > 2$, a contradiction as $e(\mathcal{H}) = n^2$. Thus, S has at least $n - n^{1-\varepsilon}$ disjoint transversals with size at least $n - n^{1-\varepsilon}$, as required. \square

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