

Hamiltonicity in random graphs is born resilient

Richard Montgomery*

Abstract

Let $\{G_M\}_{M \geq 0}$ be the random graph process, where G_0 is the empty graph on n vertices and subsequent graphs in the sequence are obtained by adding a new edge uniformly at random. For each $\varepsilon > 0$, we show that, almost surely, any graph G_M with minimum degree at least 2 is not only Hamiltonian (as shown by Bollobás), but remains Hamiltonian despite the removal of any set of edges, as long as at most $(1/2 - \varepsilon)$ of the edges incident to each vertex are removed. We say that such a graph is $(1/2 - \varepsilon)$ -resiliently Hamiltonian. Furthermore, for each $\varepsilon > 0$, we show that, almost surely, each graph G_M is not $(1/2 + \varepsilon)$ -resiliently Hamiltonian. These results strengthen those by Lee and Sudakov on the likely resilience of Hamiltonicity in the binomial random graph.

For each k , we denote by $G^{(k)}$ the (possibly empty) maximal subgraph with minimum degree at least k of a graph G . That is, the k -core of G . Krivelevich, Lubetzky and Sudakov have shown that, for each $k \geq 15$, in almost every random graph process $\{G_M\}_{M \geq 0}$, every non-empty k -core is Hamiltonian. We show that, for each $\varepsilon > 0$ and $k \geq k_0(\varepsilon)$, in almost every random graph process $\{G_M\}_{M \geq 0}$, every non-empty k -core is $(1/2 - \varepsilon)$ -resiliently Hamiltonian, but not $(1/2 + \varepsilon)$ -resiliently Hamiltonian.

1 Introduction

The appearance of Hamilton cycles in random graphs has been studied since the pioneering work by Erdős and Rényi in 1959 [10]. As Hamilton cycles, by definition, contain every vertex in their parent graph, any graph containing a Hamilton cycle has minimum degree at least 2. As is well-known, if $p = (\log n + \log \log n - \omega(1))/n$, then the binomial random graph $G(n, p)$ almost surely has a vertex with degree at most 1, and hence is not Hamiltonian. Conversely, improving on breakthrough results by Pósa [20] and Korshunov [14], in 1983 Bollobás [5] and Komlós and Szemerédi [13] independently showed that, if $p = (\log n + \log \log n + \omega(1))/n$, then $G(n, p)$ is almost surely Hamiltonian.

Suppose instead we consider the n -vertex random graph process $\{G_M\}_{M \geq 0}$, where G_0 is the graph with n vertices and no edges, and each subsequent graph G_M is formed from G_{M-1} by the addition of an edge uniformly at random, until the complete graph $G_{\binom{n}{2}}$ is formed. In 1984, Bollobás [6] showed that, in almost every random graph process, the very edge which is added to raise the minimum degree to 2 will also create a Hamilton cycle. The random graph process is strongly linked to binomial random graphs (see, for example, Section 2.1), and thus we can infer from this beautiful result the previous known results on the likely Hamiltonicity of $G(n, p)$. The result by Bollobás demonstrates that the likely obstacle to the existence of a Hamilton cycle in a random graph is the existence of some vertex of degree less than 2. In this paper, we will show that, once this obstacle is overcome, it is very likely that the graph is not only Hamiltonian, but *resiliently* Hamiltonian.

The general study of the resilience of different graph properties in the random graph was initiated by Sudakov and Vu [22], and has since seen the consideration of a variety of different properties (see, for example, [1, 2, 9]). Given a graph G satisfying a property \mathcal{P} , the *local resilience* of G with respect to \mathcal{P} is the largest integer r such that, given any graph $H \subset G$ with maximum degree at most r , the graph $G - H$ has the property \mathcal{P} . Sudakov and Vu [22] showed that, if $p > \log^4 n/n$, then the local resilience of Hamiltonicity in $G(n, p)$ is almost surely $(1/2 + o(1))pn$, and conjectured that this remains true as long

*Trinity College, Cambridge, UK, CB2 1TQ; r.h.montgomery@dpmms.cam.ac.uk

as $p = \omega(\log n/n)$. When $p = \omega(\log n/n)$, Frieze and Krivelevich [11] and Ben-Shimon, Krivelevich and Sudakov [3, 4] gave increasingly strong bounds for such likely local resilience of Hamiltonicity in $G(n, p)$, before Lee and Sudakov [16] confirmed this conjecture. More precisely, Lee and Sudakov showed that, for every $\varepsilon > 0$, there exists some constant C , such that, if $p \geq C \log n/n$, then the local resilience of $G(n, p)$ is almost surely between $(1/2 - \varepsilon)pn$ and $(1/2 + \varepsilon)pn$.

What hope have we of improving the known range of probability p for which $G(n, p)$ is likely to be resiliently Hamiltonian? As we decrease p within the range of likely Hamiltonicity, the degree sequence of $G(n, p)$ typically becomes increasingly irregular. In particular, the likely minimum degree of $G(n, p)$ will drift proportionally away from the expected average degree $(n - 1)p$. For example, when $p = (\log n + 2 \log \log n)/n$, the minimum degree of $G(n, p)$ is almost surely 2 or 3; the likely local resilience of Hamiltonicity cannot then be more than 1. The definition of local resilience here is rather weak, considering few edge sets for removal. In order to better study the resilience of Hamiltonicity in $G(n, p)$ for such values of p , the following definition is thus preferable (see also [4]).

Definition 1.1. We say a graph G is α -resilient with respect to the property \mathcal{P} if, given any subgraph $H \subset G$, with $d_H(v) \leq \alpha d_G(v)$ for each $v \in V(G)$, the graph $G - H$ has property \mathcal{P} .

If $p = \omega(\log n/n)$, then it is very likely that each vertex in $G(n, p)$ has degree $(1 + o(1))pn$. Therefore, the result by Lee and Sudakov [16] quoted above implies that, if $p = \omega(\log n/n)$, then $G(n, p)$ is almost surely $(1/2 - o(1))$ -resiliently Hamiltonian. In this paper, we extend this by proving that, in almost every random graph process, every Hamiltonian graph is $(1/2 - o(1))$ -resiliently Hamiltonian.

Theorem 1.2. Let $\varepsilon > 0$. In almost every n -vertex random graph process $\{G_M\}_{M \geq 0}$, the following is true for each $0 \leq M \leq \binom{n}{2}$. If $\delta(G_M) \geq 2$, then G_M is $(1/2 - \varepsilon)$ -resiliently Hamiltonian, but not $(1/2 + \varepsilon)$ -resiliently Hamiltonian.

Ben-Shimon, Krivelevich and Sudakov [4] used a further, more general, definition of resilience when studying the Hamiltonicity of random graphs. Let \mathbf{k} be a sequence of n integers. A graph G with the vertex set $[n]$ is said to be \mathbf{k} -resilient with respect to the property \mathcal{P} if, for any subgraph $H \subset G$ with $d_H(i) \leq \mathbf{k}(i)$, for each i , the graph $G - H$ has the property \mathcal{P} . Ben-Shimon, Krivelevich and Sudakov showed that, for every $\varepsilon > 0$ and $p = (\log n + \log \log n + \omega(1))/n$, the random graph $G = G(n, p)$ is almost surely \mathbf{k} -resiliently Hamiltonian with

$$\mathbf{k}(i) = \begin{cases} d_G(i) - 2 & \text{if } d_G(i) \leq pn/100, \\ (1/3 - \varepsilon)d_G(i) & \text{otherwise.} \end{cases}$$

The methods of this paper could be used to extend this result to use $\mathbf{k}(i) = d_G(i) - 2$ if $d_G(i) \leq pn/100$, and $\mathbf{k}(i) = (1/2 - \varepsilon)d_G(i)$ otherwise; the constant $1/2$ would then be tight. We will concentrate, however, on proving the cleaner statement of Theorem 1.2.

For large k , we will also use our methods to demonstrate that, in almost every random graph process, the k -core is born resiliently Hamiltonian. From the results mentioned above, we know that the possible obstruction to Hamiltonicity in any random graph $G(n, p)$ is almost surely vertices of degree 0 or 1. If we iteratively remove vertices of degree 0 or 1, are we likely to find a Hamiltonian subgraph? This process would find the largest subgraph with minimum degree at least 2, a structure known as the 2-core. For more general k , the k -core of a graph G , denoted $G^{(k)}$, is the (possibly empty) maximal subgraph with minimum degree at least k . This concept was introduced by Bollobás [6], who showed that, for each $k \geq 3$, there exists a constant $C(k)$ such that, if $p \geq C(k)/n$, then the k -core of $G(n, p)$ is almost surely non-empty and k -connected. Among results for more general k , Łuczak [17] showed that, if $p = (\log n + 6 \log \log n + \omega(1))/3n$, then the 2-core of $G(n, p)$ is almost surely Hamiltonian. In other words, well before the random graph is reliably Hamiltonian, it is likely that, if we iteratively remove vertices with degree 0 or 1, then the remaining graph is Hamiltonian.

For each $k \geq 3$, Bollobás, Cooper, Fenner and Frieze [8] showed that there is some $C(k)$ for which, if $p \geq C(k)/n$, then the k -core of $G(n, p)$ is almost surely Hamiltonian, where $C(k) = (2 + o_k(1))k^3$. For each $k \geq 15$, Krivelevich, Lubetzky and Sudakov [15] subsequently showed that, in almost every random

graph process, every non-empty k -core is Hamiltonian. We will show that, in almost every random graph process, every non-empty k -core is furthermore $(1/2 - o_k(1))$ -resiliently Hamiltonian, as follows.

Theorem 1.3. *For each $\varepsilon > 0$, there exists some k_0 such that, for each $k \geq k_0$, in almost every n -vertex random graph process $\{G_M\}_{M \geq 0}$, the following holds for each $0 \leq M \leq \binom{n}{2}$. If $G_M^{(k)} \neq \emptyset$, then $G_M^{(k)}$ is $(1/2 - \varepsilon)$ -resiliently Hamiltonian, but not $(1/2 + \varepsilon)$ -resiliently Hamiltonian.*

Our methods to prove Theorem 1.3 are quite different from those used by Krivelevich, Lubetzky and Sudakov in [15], and owe more to the work by Lee and Sudakov [16] on the resilience of Hamiltonicity in $G(n, p)$. The techniques we use offer an alternative proof of the result by Krivelevich, Lubetzky and Sudakov [15] that, for large k , in almost every random graph process, when the k -core is non-empty it is Hamiltonian. This alternative proof would be simpler than that found in [15], but would hold only for larger k than used there.

The heart of our paper is in the proof of a more general result, Theorem 1.6, which provides an (almost-sure) rule that determines certain induced large subgraphs of $G(n, p)$ are resiliently Hamiltonian. It is then relatively simple to confirm that the subgraphs in Theorem 1.2 and 1.3 almost surely satisfy this rule. In order to state Theorem 1.6, we require the following two definitions.

Definition 1.4. We say H is an α -residual subgraph of a graph G if, for each $v \in V(H)$, we have $d_H(v) \geq \alpha d_{G[V(H)]}(v)$.

Definition 1.5. A graph H is a 2-expander if it is connected and, for every subset $U \subset V(H)$ with $|U| \leq |H|/8$, we have $|N(U)| \geq 2|U|$.

Theorem 1.6. *For each $\varepsilon > 0$, there exists $\delta, C > 0$ such that, if $p \geq C/n$, then $G = G(n, p)$ has the following property with probability $1 - o(n^{-3})$. If H is a $(1/2 + \varepsilon)$ -residual subgraph of G , with $|H| \geq \varepsilon n$, which contains a spanning 2-expander with at most δpn^2 edges, then H is Hamiltonian.*

The constant $1/2$ appearing in Theorem 1.6 cannot be reduced. We confirm this with the following lemma.

Lemma 1.7. *For each $\varepsilon > 0$, there exists C such that, if $p \geq C/n$, then, with probability $1 - o(n^{-3})$, $G(n, p)$ contains no subset $U \subset V(G)$, with $|U| \geq \varepsilon n$, for which $G[U]$ is $(1/2 + \varepsilon)$ -resiliently Hamiltonian.*

We prove Theorem 1.6 in Section 2, where, after giving some simple properties of random graphs, we include a sketch of the proof. Lemma 1.7 is proved in Section 3. For Theorems 1.2 and 1.3, we demonstrate that large sets resiliently expand in the same manner, proving Theorem 4.2 from Theorem 1.6 in Section 4. Using Theorem 4.2 and Lemma 1.7, Theorems 1.2 and 1.3 are proved in Sections 5 and 6, respectively. In the rest of this section, we will cover our basic notation.

1.1 Notation

A graph G has vertex set $V(G)$, edge set $E(G)$, minimum degree $\delta(G)$ and maximum degree $\Delta(G)$, and $|G| = |V(G)|$. When $A \subset V(G)$, $N(A)$ is the set of neighbours of vertices in A in $V(G) \setminus A$. When $x \in V(G)$, $d(x)$ is the degree of x in G . Where multiple graphs are considered, we refer to the relevant graph in the subscript, using, for example, $d_G(x)$.

For a graph G and a vertex set $U \subset V(G)$, $G[U]$ and $G - U$ are the induced subgraphs of G with vertex sets U and $V(G) \setminus U$, respectively. For any graphs G and H , $G - H$ is the graph with vertex set $V(G)$ and edge set $E(G) \setminus E(H)$, and $G \cup H$ is the graph with vertex set $V(G) \cup V(H)$ and edge set $E(G) \cup E(H)$. For a graph G and a set $E \subset V(G)^{(2)}$, $G + E$ and $G - E$ are the graphs with vertex set $V(G)$ and edge sets $E(G) \cup E$ and $E(G) \setminus E$, respectively. For any graph G and vertex sets $A, B \subset V(G)$, $e_G(A, B)$ is the number of pairs (x, y) with $xy \in E(G)$, $x \in A$ and $y \in B$.

If $f(n)/g(n) \rightarrow 0$ as $n \rightarrow \infty$, then we say $g(n) = \omega(f(n))$ and $f(n) = o(g(n))$. If there exists a constant C for which $f(n) \leq Cg(n)$ for all n , then we say $f(n) = O(g(n))$ and $g(n) = \Omega(f(n))$. If $f = O(g(n))$ and $f(n) = \Omega(g(n))$, then we say that $f(n) = \Theta(g(n))$. The binomial random graph $G(n, p)$ has vertex set $[n] = \{1, \dots, n\}$ and edges chosen independently at random with probability p . We denote the complete graph on $[n]$ by K_n .

2 Proof of Theorem 1.6

In this section, we first cover how we move between the binomial random graph and the random graph process, and cover a few simple properties of the binomial random graph. Then, in Section 2.3, we reintroduce Pósa rotation and give a sketch of our proof of Theorem 1.6. In Sections 2.4 and 2.5 we prove two results, Corollary 2.11 and Lemma 2.12, which allow us to prove Theorem 1.6 in Section 2.6.

2.1 Model switching

We will often find it convenient to show properties hold in $G(n, p)$, before moving to the random graph process using the following standard lemma (see, for example, Bollobás [7]), where $G_{n, M}$ is chosen uniformly at random from the graphs with vertex set $[n]$ and M edges.

Lemma 2.1. *Let $n \in \mathbb{N}$, $1 \leq M \leq \binom{n}{2}$ and $p = M/\binom{n}{2}$, and let \mathcal{P} be a graph property. Then*

$$\mathbb{P}(G_{n, M} \text{ has property } \mathcal{P}) \leq 2n \cdot \mathbb{P}(G(n, p) \text{ has property } \mathcal{P}). \quad \square$$

Typically then, we will show, for any p , that a property holds in $G(n, p)$ with probability $1 - o(n^{-3})$. By Lemma 2.1 then, it holds in $G_{n, pN}$ with probability $1 - o(n^{-2})$. As, in the n -vertex random graph process $\{G_M\}_{M \geq 0}$, G_M is distributed as $G_{n, M}$, the property therefore holds throughout almost every random graph process.

2.2 Properties of the binomial random graph

The probabilistic results we need for Theorem 1.6 follow simply from Chernoff's inequality (see, for example, Janson, Łuczak and Ruciński [12, Corollary 2.3]).

Lemma 2.2. *If X is a binomial variable with standard parameters n and p , denoted $X = \text{Bin}(n, p)$, and ε satisfies $0 < \varepsilon \leq 3/2$, then*

$$\mathbb{P}(|X - \mathbb{E}X| \geq \varepsilon \mathbb{E}X) \leq 2 \exp(-\varepsilon^2 \mathbb{E}X/3). \quad \square$$

When A and B are vertex sets, the parameter $e_G(A, B)$ is close to being binomially distributed, and its typical value can be bounded using the following simple proposition.

Proposition 2.3. *Let $n \in \mathbb{N}$ and $0 \leq p \leq 1$. Suppose X_1, \dots, X_n are independent random variables, each equal to 1 with probability p , and 0 otherwise. Suppose $\delta_i \in \{1, 2\}$, $i \in [n]$, and $X = \sum_i \delta_i X_i$. Then, for each $0 < \varepsilon < 1$, we have*

$$\mathbb{P}(|X - \mathbb{E}X| \geq \varepsilon \mathbb{E}X) \leq 4 \exp(-\varepsilon^2 \mathbb{E}X/9). \quad (1)$$

Proof. Note that, if $\mathbb{E}X \leq 9p$, then the right hand side of (1) is larger than 1 and the result is trivial. Assume that $\mathbb{E}X \geq 9p$. For each $i \in [n]$, pick $\delta_{i,1}, \delta_{i,2} \in \{0, 1\}$ so that $\delta_i = \delta_{i,1} + \delta_{i,2}$ and, if $Y_1 = \sum_i \delta_{i,1} X_i$ and $Y_2 = \sum_i \delta_{i,2} X_i$, then $|\mathbb{E}Y_1 - \mathbb{E}Y_2| \leq p$. As $\mathbb{E}X \geq 9p$, we have $\mathbb{E}Y_1, \mathbb{E}Y_2 \geq \mathbb{E}X/3$. By Lemma 2.2, $\mathbb{P}(|Y_1 - \mathbb{E}Y_1| \geq \varepsilon \mathbb{E}Y_1) \leq 2 \exp(-\varepsilon^2 \mathbb{E}Y_1/3) \leq 2 \exp(-\varepsilon^2 \mathbb{E}X/9)$. A similar result holds for Y_2 , so that

$$\mathbb{P}(|X - \mathbb{E}X| \geq \varepsilon \mathbb{E}X) \leq \mathbb{P}(|Y_1 - \mathbb{E}Y_1| \geq \varepsilon \mathbb{E}Y_1) + \mathbb{P}(|Y_2 - \mathbb{E}Y_2| \geq \varepsilon \mathbb{E}Y_2) \leq 4 \exp(-\varepsilon^2 \mathbb{E}X/9). \quad \square$$

Using Proposition 2.3, we can give a simple bound on the number of edges we can expect between any two large sets in $G(n, p)$, as follows.

Lemma 2.4. *Let $\varepsilon > 0$ and $G = G(n, p)$. With probability $1 - o(n^{-3})$, if $A, B \subset V(G)$ and $p|A||B| \geq 100n/\varepsilon^2$, then $(1 - \varepsilon)p|A||B| \leq e(A, B) \leq (1 + \varepsilon)p|A||B|$.*

Proof. By Proposition 2.3, the property in the lemma does not hold with probability at most

$$\sum_{A, B \subset V(G), p|A||B| \geq 100n/\varepsilon^2} 4 \exp(-\varepsilon^2 p|A||B|/9) \leq 2^{2n} \cdot 4 \exp(-10n) = o(n^{-3}). \quad \square$$

It will be useful to have a bound on the expected number of subgraphs of $G(n, p)$ with at most δpn^2 edges, for any small fixed δ . For this, we will use the following proposition.

Proposition 2.5. *For each $0 < \delta < 1$, there exists n_0 such that, for each $n \geq n_0$ and $p \geq 1/n$,*

$$\sum_{H \subset K_n, e(H) \leq \delta pn^2} \mathbb{P}(H \subset G(n, p)) \leq \exp(2\delta \log(e/\delta)pn^2).$$

Proof. We have

$$\begin{aligned} \sum_{H \subset K_n, e(H) \leq \delta pn^2} \mathbb{P}(H \subset G(n, p)) &\leq \sum_{i=0}^{\delta pn^2} \binom{n^2}{i} p^i \leq \sum_{i=0}^{\delta pn^2} \left(\frac{epn^2}{i}\right)^i \leq n^2 \cdot (e/\delta)^{\delta pn^2} \\ &\leq \exp(2\delta \log(e/\delta)pn^2), \end{aligned}$$

where the last inequalities hold for sufficiently large $n \geq n_0$. \square

2.3 Pósa rotation and proof sketch

The rotation-extension technique was first introduced by Pósa [20] to study the threshold for Hamiltonicity in the random graph, and has since been used in the proofs of many results concerning Hamilton cycles. Given a path $x_1x_2 \dots x_k$ and an edge x_kx_j , for some $j < k - 1$, a *rotation with x_1 fixed* is made by *breaking* the edge x_jx_{j+1} and considering the new path $x_1 \dots x_jx_kx_{k-1} \dots x_{j+1}$. Thus, we find a new path with the same vertex set but which starts at x_1 and ends at $x_{j+1} \neq x_k$. If we are able to rotate both ends of the path multiple times, then we can find many pairs of vertices, such that, if any one of the pairs is an edge, then there is a cycle with the same vertex set as the path.

As shown by Pósa [20], if a graph H is a 2-expander, then we can rotate a maximal length path in H many times to find other maximal length paths with a different endvertex, to get the following lemma.

Lemma 2.6. *If H is a 2-expander and $U \subset V(H)$ supports a maximal length path in H with endvertex $v \in U$, then there are at least $|H|/8$ vertices $u \in U$ for which there is a u, v -Hamilton path in $H[U]$. \square*

If any of the vertex pairs $u, v \in U$ in Lemma 2.6 are added to $E(H)$ then H contains a cycle with length $|U|$. Furthermore, if H is connected and $|U| < |H|$, by considering a neighbour of this cycle we can find a path with $|U| + 1$ vertices. We will start with a sparse 2-expander in our random graph and add edges which increase the maximum length of a path in the subgraph, or make it Hamiltonian. To describe this, we use the following definition.

Definition 2.7. In a graph H , we say $E \subset V(H)^{(2)}$ is a *booster for H* if $H + E$ contains a longer path than H does, or $H + E$ is Hamiltonian. If $e \in V(H)^{(2)}$, and $\{e\}$ is a booster for H , then we also say that e is a booster for H .

Note that, if we iteratively add $|H|$ boosters to H , then, as the length of the maximum path is at most $|H| - 1$, the resulting graph must be Hamiltonian.

A standard method to find a Hamilton cycle in $G(n, (\log n + \log \log n + \omega(1))/n)$ runs as follows. Letting $G_0 = G(n, (\log n + \log \log n + \omega(1))/n)$, we can easily show that G_0 is almost surely a 2-expander. Therefore, from Lemma 2.6 applied twice to a maximal length path in G_0 , $V(G_0)^{(2)}$ contains at least $n^2/128$ boosters for G_0 . Revealing more edges with probability $10^3/n^2$ (say) to get G_1 , with probability at least $1/2$, G_0 has some booster in G_1 . Repeating this $k = \omega(n)$ times to get $G_0 \supset G_1 \supset \dots \supset G_k$, we almost surely have at least n values for i for which $G_i - G_{i-1}$ contains a booster for G_{i-1} , and hence G_k is Hamiltonian. Note that in total each edge has been revealed with probability $(\log n + \log \log n + \omega(1))/n$. This method is known as *sprinkling*, but cannot withstand the later removal of edges. We need therefore new methods to show that random graphs are *resiliently* Hamiltonian.

From the work by Lee and Sudakov [16] on the resilience of Hamiltonicity in $G(n, \omega(\log n/n))$, we can learn the following principle: almost surely, if we have any sparse subgraph $H_0 \subset G = G(n, \omega(1/n))$

and desire one of $\Omega(n^2)$ possible edges to exist in $G - H_0$, then some such edge does exist in $G - H_0$. In this, the desired edges in $G - H_0$ can be determined by H_0 . The principle comes, as for each possible sparse subgraph H_0 , it is far more likely that at least one of the $\Omega(n^2)$ desired edges exists in $G - H_0$ than that $H_0 \subset G$, so much so that this can overpower the possible number of sparse subgraphs H_0 . A calculation along this line appears after Claim 2.9 in the proof of Lemma 2.8.

From the work by Lee and Sudakov [16], with only slight modification we have the following method to find a Hamilton cycle in $G(n, \omega(\log n/n))$. It is easy to show that $G(n, 2 \log n/n)$ is almost surely a 2-expander, and therefore $G = G(n, \omega(\log n/n))$ almost surely contains a sparse 2-expander H_0 . From the principle above and Lemma 2.6, we can show that, almost surely, for any sparse 2-expander $H \subset G$, $G - H$ contains a booster edge for H . Starting then, with H_0 , we can iteratively add booster edges to H_0 until it is Hamiltonian, where the graph remains relatively sparse as at most n booster edges are added. Importantly, this basic technique can be made to withstand the later removal of edges. As Lee and Sudakov showed, a careful analysis reveals that, for sparse 2-expanders H_0 , the possibilities for booster edges in $G - H_0$ are so numerous that almost surely some must lie in $H - H_0$ for any $(1/2 + \varepsilon)$ -residual subgraph $H \subset G$. Furthermore, almost surely, any such subgraph $H \subset G$ can be shown to contain a spanning sparse 2-expander, so that this argument can be used to show that $G(n, \omega(\log n/n))$ is almost surely $(1/2 - \varepsilon)$ -resiliently Hamiltonian.

We have already made a small conceptual change to Lee and Sudakov's argument to allow us to use a lower edge probability. The main novelty, however, in our methods is to use the principle outlined above twice. Essentially, we use it to show that the number of potential booster sets with at most two edges for any sparse 2-expander $H_0 \subset G$ is almost surely very large, before using it again to show that some of these potential booster sets almost surely lie in any $(1/2 + \varepsilon)$ -residual subgraph H of G . These two steps are carried out in Sections 2.4 and 2.5, respectively, before Theorem 1.6 is proved in Section 2.6.

2.4 Resilient rotation

We will work within a random graph $G = G(n, p)$ with a sparse 2-expander $H_0 \subset G$ and a $(1/2 + \varepsilon)$ -residual subgraph $H \subset G$ with $V(H) = V(H_0)$. We will show that there are a linear number of vertices v in H_0 for which there are $(1/2 + \varepsilon)|H|$ vertices $u \in V(H_0)$ for which we can find a set $E_{v,u}$ such that, for each $e \in E_{v,u}$, uv is a booster for $H_0 + e$ (see Lemma 2.8). Taking a random sparse subgraph $H_1 \subset G - H$ typically retains some edge in enough of the sets $E_{v,u}$ that we may show there is some sparse subgraph $H_1 \subset G - H$ for which we have many pairs v, u such that there is some edge $e \in E(H_1)$ for which uv is a booster for $H_0 + e$ (thus proving Corollary 2.11).

We cannot simply add H_1 to H_0 when it is found, as on iterative application this would increase the size of H_0 too much. It is important that at each application we add two edges (one from H_1 and one from $H - H_0 - H_1$) to get a longer path or Hamilton cycle in H_0 , so that H_0 is kept relatively sparse in comparison to G .

Lemma 2.8. *For each $0 < \varepsilon \leq 1$ there exists $\delta, C > 0$ such that, if $p \geq C/n$, then $G = G(n, p)$ has the following property with probability $1 - o(n^{-3})$. Let $H_0 \subset G$ be a 2-expander with at least εn vertices and at most $2\delta pn^2$ edges and let H be a $(1/2 + \varepsilon)$ -residual subgraph of G with $V(H) = V(H_0)$. Then, for at least $|H|/8$ vertices $v \in V(H)$ there is some set $U_v \subset V(H)$ with $|U_v| \geq (1/2 + \varepsilon/8)|H|$ and disjoint subsets $E_{v,u} \subset E(H - H_0)$, $u \in U_v$, so that $|E_{v,u}| \geq 50/\varepsilon\delta$ and, for each $u \in U_v$ and $e \in E_{v,u}$, $\{uv, e\}$ is a booster for H_0 .*

Proof. Let $\delta, C > 0$ be determined later. Let \mathcal{H} be the set of all 2-expander graphs H_0 with $V(H_0) \subset [n]$, $e(H_0) \leq 2\delta pn^2$ and $|H_0| \geq \varepsilon n$. Let $H_0 \in \mathcal{H}$. By Lemma 2.6, there are at least $|H_0|/8$ vertices v , say those in V_{H_0} , which appear at the end of a longest path in H_0 .

For each $v \in V_{H_0}$, let $f(H_0, v)$ be the event that, for each $(1/2 + \varepsilon)$ -residual subgraph H of $G = G(n, p)$ with $V(H) = V(H_0)$, there is some set $U \subset V(H)$ with $|U| \geq (1/2 + \varepsilon/8)|H|$ and disjoint subsets $E_u \subset E(H - H_0)$, $u \in U$, so that $|E_u| \geq 50/\varepsilon\delta$ and, for each $u \in U$ and $e \in E_u$, $\{uv, e\}$ is a booster for H_0 .

Claim 2.9. For each $H_0 \in \mathcal{H}$ and $v \in V_{H_0}$, $\mathbb{P}(\overline{f(H_0, v)}) \leq \exp(-\varepsilon^5 pn^2/10^6)$.

From this claim, we can show, as follows, that, with probability $1 - o(n^{-3})$, $f(H_0, v)$ holds for each $H_0 \in \mathcal{H}$ and $v \in V_{H_0}$ with $H_0 \subset G$. Note that the events $f(H_0, v)$ and $H_0 \subset G$ are independent. The probability that $f(H_0, v)$ does not hold and $H_0 \subset G$ for some $H_0 \in \mathcal{H}$ and $v \in V_{H_0}$ is then at most

$$\begin{aligned} \sum_{H_0 \in \mathcal{H}} \sum_{v \in V_{H_0}} \mathbb{P}(\overline{f(H_0, v)}) \cdot \mathbb{P}(H_0 \subset G) &\leq n \cdot \exp(-\varepsilon^5 pn^2/10^6) \cdot \sum_{H_0 \subset K_n, e(H_0) \leq 2\delta pn^2} \mathbb{P}(H_0 \subset G) \\ &\leq n \cdot \exp(-(\varepsilon^5/10^6 - 2\delta \log(e/2\delta))pn^2), \end{aligned}$$

where we have used Proposition 2.5. Therefore, we can choose δ to be sufficiently small, depending only on ε , for this to hold with probability $o(n^{-3})$. Thus, we need only prove Claim 2.9.

Proof of Claim 2.9. Let $H_0 \in \mathcal{H}$ and $v \in V_{H_0}$. Let k be the length of a longest path in H_0 . Pick some path $P \subset H_0$ with length k and v as an end-vertex. By Lemma 2.6, as H_0 is a 2-expander, we can find a set $A \subset V(P)$ with $|A| = \varepsilon|H_0|/30$ so that, for each $a \in A$, there is a v, a -path, P_a say, in $H_0[V(P)]$ with length k .

Let $B = V(H_0) \setminus A$. For each $u \in B \cap V(P)$, let X_u be the set of pairs $\{a, b\}$ with $a \in A$ and $b \in V(P) \setminus A$ so that P_a could be rotated in $H_0 + ab$ with v fixed, using the edge ab , to get u as a new endpoint. Note that, here, $\{uv, ab\}$ is a booster for H_0 , and the sets X_u , $u \in B \cap V(P)$ are disjoint and satisfy $|X_u| \leq |A|$.

For each $u \in B \setminus V(P)$, let X_u be the set of pairs $\{u, a\}$ with $a \in A$. Note that, here, $\{ua\}$, and hence $\{uv, ua\}$, is a booster for H_0 . For each $u \in B$ and $e \in X_u$, we have that $\{uv, e\}$ is a booster for H_0 . Note further that the sets X_u , $u \in B$, are disjoint and contain vertex pairs with exactly one vertex in A , and, for each $u \in B$, $|X_u| \leq |A|$.

For each $a \in A$ and each $b \in V(P) \setminus A$, P_a can be rotated using ab to get a new endvertex, at most $|A|$ of which can be in A . Therefore, $|\cup_{u \in B \cap V(P)} X_u| \geq |A|(|P| - 2|A|)$. Note that $|\cup_{u \in B \setminus V(P)} X_u| = |A|(|H_0| - |P|)$. For each $u \in B$, let $Y_u = X_u \setminus E(H_0)$. Then,

$$|\cup_{u \in B} Y_u| \geq |A|(|P| - 2|A|) + |A|(|H_0| - |P|) - 2\delta pn^2 = |A||H_0| - 2|A|^2 - 2\delta pn^2 \geq (1 - \varepsilon/8)|A||H_0|,$$

for sufficiently small δ . For each $u \in B$, we have, furthermore, that $|Y_u| \leq |X_u| \leq |A|$. For each $a \in A$, let Z_a be the pairs in $\cup_{u \in B} Y_u$ which contain a , so that $\cup_{a \in A} Z_a = \cup_{u \in B} Y_u$, and, hence, $|\cup_{a \in A} Z_a| = |\cup_{u \in B} Y_u| \geq (1 - \varepsilon/8)|A||H_0|$.

Define the events F_1 , F_2 , and F_3 , as follows.

$$F_1: |\cup_{a \in A} (Z_a \cap E(G))| \geq (1 - \varepsilon/4)p|A||H_0|.$$

$$F_2: \sum_{a \in A} d_{G[V(H_0)]-H_0}(a) \leq (1 + \varepsilon/8)p|A||H_0|.$$

$$F_3: \text{If } U \subset B \text{ with } |U| \leq (1/2 + \varepsilon/8)|H_0|, \text{ then } |\cup_{u \in U} (Y_u \cap E(G))| < (1/2 + \varepsilon/4)p|A||H_0|.$$

As $|A||H_0| \geq \varepsilon^3 n^2/30$, and $\mathbb{E}|\cup_{a \in A} (Z_a \cap E(G))| \geq (1 - \varepsilon/8)p|A||H_0|$, by Lemma 2.2, we have $\mathbb{P}(\overline{F_1}) \leq \exp(-\varepsilon^5 pn^2/10^5)$.

Furthermore, using Proposition 2.3, and as $|A||H_0| \geq \varepsilon^3 n^2/30$, we have

$$\mathbb{P}(\overline{F_2}) \leq \mathbb{P}\left(\sum_{a \in A} d_{G[V(H_0)]-H_0}(a) > (1 + \varepsilon/8)p|A||H_0|\right) \leq 4 \exp(-\varepsilon^5 pn^2/10^5).$$

Finally, for each $U \subset B$ with $|U| \leq (1/2 + \varepsilon/8)|H_0|$, we have $\mathbb{E}|\cup_{u \in U} (Y_u \cap E(G))| \leq (1/2 + \varepsilon/8)p|A||H_0|$. Therefore, using Proposition 2.3, as $|A||H_0| \geq \varepsilon^3 n^2/30$, we have

$$\mathbb{P}(\overline{F_3}) \leq 2^n \cdot 4 \exp(-\varepsilon^5 pn^2/10^5) \leq \exp(-\varepsilon^5 pn^2/(2 \cdot 10^5)),$$

for sufficiently large C as $pn^2 \geq Cn$. Therefore, $\mathbb{P}(F_1 \wedge F_2 \wedge F_3) \geq 1 - \exp(-\varepsilon^5 pn^2/10^6)$. We will show that, if F_1 , F_2 , and F_3 hold, then $f(H_0, v)$ holds, completing the proof of the claim.

Suppose then that F_1 , F_2 , and F_3 hold, and let H be a $(1/2 + \varepsilon)$ -residual subgraph of G with $V(H) = V(H_0)$. For each $u \in B$, let $E_u = Y_u \cap E(H)$, so that, for each $e \in E_u$, $\{uv, e\}$ is a booster for H_0 . Let U be the set of vertices in B for which $|E_u| \geq 50/\varepsilon\delta$. We need only show that $|U| \geq (1/2 + \varepsilon/8)|H_0|$.

Now, taking δ to be small, and then, depending on δ , C to be large,

$$\begin{aligned}
|\cup_{u \in U} (Y_u \cap E(G))| &\geq |\cup_{u \in B} (Y_u \cap E(G))| - |\cup_{u \in B \setminus U} (Y_u \cap E(G - H))| - \sum_{u \in B \setminus U} |E_u| \\
&\geq |\cup_{a \in A} (Z_a \cap E(G))| - |\cup_{a \in A} (Z_a \cap E(G - H))| - (50/\varepsilon\delta)|B \setminus U| \\
&\stackrel{F_1}{\geq} (1 - \varepsilon/4)p|A||H_0| - \sum_{a \in A} (1/2 - \varepsilon)d_{G[V(H_0)]}(a) - 50|H_0|/\varepsilon\delta \\
&\stackrel{F_2}{\geq} (1 - \varepsilon/4)p|A||H_0| - (1/2 - \varepsilon)(1 + \varepsilon/8)p|A||H_0| - 2e(H_0) - 10^4p|A||H_0|/\varepsilon^3\delta C \\
&\geq (1/2 - \varepsilon/4 + \varepsilon - \varepsilon/8 - 120\delta/\varepsilon^3 - 10^4/\varepsilon^3\delta C)p|A||H_0| \\
&\geq (1/2 + \varepsilon/4)p|A||H_0|.
\end{aligned}$$

Therefore, as F_3 holds, we have $|U| \geq (1/2 + \varepsilon/8)|H_0|$. This completes the proof of the claim, and hence the lemma. $\square \square$

Lemma 2.8 implies that there are likely to be enough boosters in H that there is a sparse subgraph of H which still contains many boosters. This is shown for Corollary 2.11, for which we use the following definition.

Definition 2.10. Given two graphs H_0 and H_1 which are edge disjoint but have the same vertex set, we say H_0 has ε -many boosters with help from H_1 if there are at least $\varepsilon|H_0|$ vertices $v \in V(H_0)$ for which there are at least $(1/2 + \varepsilon)|H_0|$ many vertices $u \in V(H_0) \setminus \{v\}$ for which there exists an $e \in E(H_0) \cup E(H_1)$ so that $\{uv, e\}$ is a booster for H_0 .

Corollary 2.11. For each $0 < \varepsilon \leq 1$ there exists $\delta, C > 0$ such that, if $p \geq C/n$, then $G = G(n, p)$ has the following property with probability $1 - o(n^{-3})$. Let $H_0 \subset G$ be a 2-expander with at least εn vertices and at most $2\delta pn^2$ edges and let H be a $(1/2 + \varepsilon)$ -residual subgraph of G with $V(H) = V(H_0)$. Then, there is some subgraph $H_1 \subset H - H_0$ with $e(H_1) \leq 2\delta pn^2$ so that H_0 has $(\varepsilon/16)$ -many boosters with help from H_1 .

Proof. Let $\delta, C > 0$ be constants for which Lemma 2.8 holds with ε . With probability $1 - o(n^{-3})$, G has the property from Lemma 2.8 and, using Lemma 2.2, at most pn^2 edges. Suppose that H_0 is a 2-expander with at least εn vertices and at most $2\delta pn^2$ edges and that H is a $(1/2 + \varepsilon)$ -residual subgraph of G with $V(H) = V(H_0)$. By the property from Lemma 2.8, we can find a set V of $|H|/8$ vertices $v \in V(H)$ for which there is some set $U_v \subset V(H)$ with $|U_v| \geq (1/2 + \varepsilon/8)|H|$ and disjoint subsets $E_{uv} \subset E(H - H_0)$, $u \in U_v$, so that $|E_{uv}| \geq 50/\varepsilon\delta$ and, for each $u \in U_v$ and $e \in E_{uv}$, $\{uv, e\}$ is a booster for H_0 .

Let H_1 be a random subgraph of $H - H_0$ with edges chosen independently at random with probability δ . By Lemma 2.2, almost surely, $e(H_1) \leq 2\delta pn^2$. For each $v \in V$, let $U'_v \subset U_v$ be the set of vertices u for which $E_{uv} \cap E(H_1) \neq \emptyset$, so that $\mathbb{P}(u \notin U'_v) \leq (1 - \delta)^{50/\varepsilon\delta} \leq \exp(-50/\varepsilon) \leq \varepsilon/32$. Therefore, as the sets E_{uv} , $u \in U_v$, are disjoint, we have, by Lemma 2.2,

$$\mathbb{P}(|U'_v| \geq (1/2 + \varepsilon/16)|H_0|) \geq \mathbb{P}(|U'_v| \geq (1 - \varepsilon/32)|U_v|) = o(n^{-1}).$$

Therefore, some such graph $H_1 \subset H$ with $e(H_1) \leq 2\delta pn^2$ exists with $|U'_v| \geq (1/2 + \varepsilon/16)|H_0|$ for each $v \in V$. \square

2.5 Finding many boosters in H

Corollary 2.11 demonstrates that there are likely to be many boosters for a sparse expander $H_0 \subset G$ with help from some other sparse subgraph H_1 . We now show that many of these boosters are likely to exist in G , sufficiently many that some exist in any $(1/2 + \varepsilon)$ -residual subgraph H of G .

Lemma 2.12. *Let $0 < \varepsilon \leq 1$. There exists $\delta, C > 0$ such that the random graph $G = G(n, p)$, with $p \geq C/n$, has the following property with probability $1 - o(n^{-3})$. Suppose H_0 and H_1 are edge-disjoint subgraphs of G with $V(H_0) = V(H_1)$, $e(H_0), e(H_1) \leq 2\delta pn^2$ and $|H_0| \geq \varepsilon n$, and where H_0 has ε -many boosters with help from H_1 . Then, for any $1/2$ -residual subgraph $H \subset G$ with $V(H) = V(H_0)$ there is some $e_1, e_2 \in E(H) \cup E(H_1)$ such that $\{e_1, e_2\}$ is a booster for H_0 .*

Proof. Let $\delta, C > 0$ be determined later, where δ will be taken to be small depending on ε and C to be large depending on δ . Let \mathcal{H} be the set of pairs (H_0, H_1) where H_0 and H_1 are edge-disjoint subgraphs with $V(H_0) = V(H_1) \subset [n]$, $e(H_0), e(H_1) \leq 2\delta pn^2$, and $|H_0| \geq \varepsilon n$, and where H_0 has ε -many boosters with help from H_1 .

Fix $(H_0, H_1) \in \mathcal{H}$, and, for each $x \in V(H_0)$, let V_x be the set of vertices $v \in V(H_0) \setminus \{x\}$ for which there is some edge $e \in H_1$ so that $\{xv, e\}$ is a booster for H_0 . Let $X = \{x \in V(H_0) : |V_x| \geq (1/2 + \varepsilon)|H_0|\}$, so that, by Definition 2.10, $|X| \geq \varepsilon|H_0| \geq \varepsilon^2 n$. Let $p \geq C/n$ and $G = G(n, p)$, and let $f(H_0, H_1)$ be the event that $\sum_{x \in X} d_{G-H_0-H_1}(x, V_x) \geq (1/2 + \varepsilon/4)p|X||H_0|$. Note that

$$\begin{aligned} \mathbb{E}\left(\sum_{x \in X} d_{G-H_0-H_1}(x, V_x)\right) &\geq (1/2 + \varepsilon)p|X||H_0| - 2e(H_0) - 2e(H_1) \\ &\geq (1/2 + \varepsilon/2)p|X||H_0| \geq \varepsilon^3 pn^2/2. \end{aligned}$$

Therefore, for C sufficiently large, by Proposition 2.3, we have

$$\mathbb{P}(\overline{f(H_0, H_1)}) \leq 4 \exp(-(\varepsilon/4)^2 \cdot \varepsilon^3 pn^2/18) \leq \exp(-\varepsilon^5 pn^2/10^3). \quad (2)$$

Note that the events $f(H_0, H_1)$, $H_0 \subset G$ and $H_1 \subset G$ are independent. Therefore, the probability that $f(H_0, H_1)$ does not hold and $H_0, H_1 \subset G$ for some $(H_0, H_1) \in \mathcal{H}$ is at most

$$\begin{aligned} \sum_{(H_0, H_1) \in \mathcal{H}} \mathbb{P}(\overline{f(H_0, H_1)} \wedge H_0, H_1 \subset G) &= \sum_{(H_0, H_1) \in \mathcal{H}} \mathbb{P}(\overline{f(H_0, H_1)}) \mathbb{P}(H_0 \subset G) \mathbb{P}(H_1 \subset G) \\ &\stackrel{(2)}{\leq} \exp(-\varepsilon^5 pn^2/10^3) \cdot \left(\sum_{H_0 \subset K_n, e(H_0) \leq 2\delta pn^2} \mathbb{P}(H_0 \subset G) \right)^2 \\ &\leq \exp(-(\varepsilon^5/10^3 - 4\delta \log(e/2\delta))pn^2), \end{aligned}$$

using Proposition 2.5. Therefore, choosing δ sufficiently small, and C sufficiently large, as $p \geq C/n$, with probability $1 - o(n^{-3})$, $f(H_0, H_1)$ holds for every $(H_0, H_1) \in \mathcal{H}$ with $H_0, H_1 \subset G$.

By this, and Lemma 2.4, with probability $1 - o(n^{-3})$ we can assume that $f(H_0, H_1)$ holds for every $(H_0, H_1) \in \mathcal{H}$ with $H_0, H_1 \subset G$, and, if $X, U \subset V(G)$, $|X| \geq \varepsilon^2 n$, and $|U| \geq \varepsilon n$, then $e_G(X, U) \leq (1 + \varepsilon/6)p|X||U|$.

Now, take any $(H_0, H_1) \in \mathcal{H}$ with $H_0, H_1 \subset G$. Using the notation for a fixed $(H_0, H_1) \in \mathcal{H}$ above, as $f(H_0, H_1)$ holds,

$$\sum_{x \in X} (d_{G-H_0-H_1}(x, V_x) - d_{G[V(H_0)]}(x)/2) \geq (1/2 + \varepsilon/4)p|X||H_0| - e_G(X, V(H_0))/2 > 0.$$

Thus, there is some $x \in V(H_0)$ with $d_G(x, V_x) > d_{G[V(H_0)]}(x)/2$. For any $1/2$ -residual subgraph $H \subset G$ with $V(H) = V(H_0)$, then, we have $d_H(x, V_x) \geq d_G(x, V_x) - d_{G[V(H_0)]}(x)/2 > 0$. That is, there must be some $v \in N_H(x) \cap V_x$. By the definition of V_x at the start of the proof, there is some $e \in E(H_1)$ such that $\{e, xv\}$ is a booster for H_0 . As this holds for each $(H_0, H_1) \in \mathcal{H}$ with $H_0, H_1 \subset G$ and each $1/2$ -residual subgraph $H \subset G$ with $V(H) = V(H_0)$, this completes the proof. \square

2.6 Proof of Theorem 1.6

Armed with Corollary 2.11 and Lemma 2.12, we can now prove Theorem 1.6.

Proof of Theorem 1.6. Let $\delta, C > 0$ be such that Corollary 2.11 holds for ε , Lemma 2.12 holds for $\varepsilon/16$, and, furthermore, let C be sufficiently large that $C\delta \geq 2$. Letting $p \geq C/n$, then, with probability $1 - o(n^{-3})$, $G = G(n, p)$ has the property in Corollary 2.11 with ε and the property in Lemma 2.12 with $\varepsilon/16$. Let H be a $(1/2 + \varepsilon)$ -residual subgraph of G with $|H| \geq \varepsilon n$ which contains a spanning 2-expander, H_0 say, with at most δpn^2 edges.

For each $1 \leq i \leq n$, find $e_{i,1}, e_{i,2} \in E(H)$ such that $\{e_{i,1}, e_{i,2}\}$ is a booster for H_{i-1} , and let $H_i = H_{i-1} + e_{i,1} + e_{i,2}$. This is possible, for each $1 \leq i \leq n$, as follows. Noting that $e(H_{i-1}) \leq \delta pn^2 + 2n \leq 2\delta pn^2$, by the property from Corollary 2.11, there is some subgraph $H' \subset H - H_{i-1}$ with $e(H') \leq 2\delta pn^2$ so that H_{i-1} has $(\varepsilon/16)$ -many boosters with help from H' . Therefore, from the property from Lemma 2.12 there is some $e_{i,1}, e_{i,2} \in E(H)$ such that $\{e_{i,1}, e_{i,2}\}$ is a booster for H_{i-1} , as required.

We have added n boosters to H_0 to get $H_n \subset H$, and therefore H_n , and hence H , is Hamiltonian. \square

3 Proof of Lemma 1.7

We will prove Lemma 1.7, for each $\varepsilon > 0$, by showing that, in a typical linear vertex subset U of a binomial random graph $G = G(n, p)$, $G[U]$ can be made into an unbalanced bipartite graph without deleting any more than $(1/2 + \varepsilon)$ of the edges around any vertex.

Proof of Lemma 1.7. Note that we may assume $0 < \varepsilon < 1/8$, and let $C = 10^3/\varepsilon^7$. By Lemma 2.4, if $p \geq C/n$, then with probability $1 - o(n^{-3})$, $G = G(n, p)$ has the following property.

P If $A, B \subset V(G)$ and $|A||B| \geq \varepsilon^5 n^2$, then $(1 - \varepsilon/3)p|A||B| \leq e_G(A, B) \leq (1 + \varepsilon/3)p|A||B|$.

Now, let $U \subset V(G)$ satisfy $|U| \geq \varepsilon n$. We will show that $G[U]$ is not $(1/2 + \varepsilon)$ -resiliently Hamiltonian. Pick a vertex partition $A \cup B = U$, so that $e_G(A, B)$ is maximised. For each $v \in A$, $d_G(v, A) \leq d_G(v, B)$, otherwise, moving v from A into B would increase $e_G(A, B)$. Similarly, for each $v \in B$, $d_G(v, B) \leq d_G(v, A)$. Therefore, if $H = G[A] \cup G[B]$, then, for each $v \in V(G)$, $d_H(v) \leq d_{G[U]}(v)/2$. If $|A| \neq |B|$, then, as $G[U] - H$ is an unbalanced bipartite graph, $G[U] - H$ is not Hamiltonian, and thus, if $|A| \neq |B|$, $G[U]$ is not $(1/2)$ -resiliently Hamiltonian.

Suppose then that $|A| = |B|$. Let $U_0 \subset U$ be the set of vertices with degree at most $1/\varepsilon$ in $G[U]$.

Claim 3.1. $|U_0| < \varepsilon^3 n$.

Proof of Claim 3.1. Suppose, for contradiction, that we may take a set $U' \subset U_0$ with $|U'| = \varepsilon^3 n$. There are at most $|U'|/\varepsilon = \varepsilon^2 n$ edges between $|U'|$ and $|U \setminus U'|$. However, as $|U'| |U \setminus U'| \geq \varepsilon^3 n \cdot |U|/2 \geq \varepsilon^5 n^2$, by **P**, there are at least $\varepsilon^5 pn^2/2 > \varepsilon^2 n$ edges between $|U'|$ and $|U \setminus U'|$, a contradiction. \square

From Claim 3.1, we have that $|U_0 \cup N_{G[U]}(U_0)| \leq 2\varepsilon^2 n$. Let $A_0 = A \setminus (U_0 \cup N_{G[U]}(U_0))$. As $|A| = |B|$, we have $|A|, |B| \geq \varepsilon n/2$, and hence $|A_0| \geq \varepsilon n/4$. Using **P**, and as $|A| = |B|$,

$$\begin{aligned} \sum_{x \in A_0} d(x, B) - (1 + \varepsilon) \sum_{x \in A_0} d(x, A) &= e(A_0, B) - (1 + \varepsilon) e(A_0, A) \\ &\leq (1 + \varepsilon/3) |A_0||B| - (1 + \varepsilon) (1 - \varepsilon/3) |A_0||A| \leq 0. \end{aligned}$$

Therefore, there exists $x \in A_0$ with $d(x, B) \leq (1 + \varepsilon)d(x, A)$, so that $d(x, B) \leq (1 + \varepsilon)d_{G[U]}(x)/2$. Let $A' = A \setminus \{x\}$ and $B' = B \cup \{x\}$ be a new partition of U . Take $H' = G[A'] \cup G[B']$, so that, by the choice of x , $d_{H'}(x) \leq (1 + \varepsilon)d_{G[U]}(x)/2$. For each vertex $v \in U \setminus N_{G[U]}(x)$, with $v \neq x$, we did not change where its neighbours lie in the partition, so $d_{H'}(v) \leq d_{G[U]}(v)/2$. For each vertex $v \in N_{G[U]}(x)$, we have $v \notin U_0$ by the choice of A_0 , so that $d_{G[U]}(v) \geq 1/\varepsilon$, and we moved one of its neighbours, x , across the partition. Thus, $d_{H'}(v) \leq d_{G[U]}(v)/2 + 1 \leq (1/2 + \varepsilon)d_{G[U]}(v)$. As $|A'| \neq |B'|$, $G[U] - H'$ is an unbalanced bipartite graph, and hence not Hamiltonian. Thus, $G[U]$ is not $(1/2 + \varepsilon)$ -resiliently Hamiltonian. \square

4 Resilient large set expansion

Set against small set expansion, it is comparatively straightforward to show that our random graphs resiliently contain a sparse subgraph in which large sets expand. We show this for both Theorem 1.2 and Theorem 1.3, through the following definition and theorem.

Definition 4.1. A graph H is a $(2, k)$ -expander if, for every subset $U \subset V(H)$, with $|U| \leq k$, we have $|N(U)| \geq 2|U|$.

Theorem 4.2. For each $\varepsilon > 0$, there exists $\delta, C > 0$ such that, if $p \geq C/n$, then $G = G(n, p)$ has the following property with probability $1 - o(n^{-3})$. Any $(1/2 + \varepsilon)$ -residual subgraph H of G , with $|H| \geq \varepsilon n$, which contains a spanning $(2, C/p)$ -expander with at most δpn^2 edges is Hamiltonian.

Proof. By Theorem 1.6, we may take $\delta > 0$ and $C \geq 200/\varepsilon^3\delta$ such that, if $p \geq C/n$ then we have the following property with probability $1 - o(n^{-3})$.

Q1 Any $(1/2 + \varepsilon)$ -residual subgraph $H \subset G$, with $|H| \geq \varepsilon n$, which contains a spanning 2-expander with at most $2\delta pn^2$ edges is Hamiltonian.

Let $p \geq C/n$ and $G = G(n, p)$, so that with probability $1 - o(n^{-3})$ we have both **Q1** and, by Lemma 2.4, the following property.

Q2 If $A, B \subset V(G)$ satisfy $|A| \geq C/p$ and $|B| \geq \varepsilon n/2$, then

$$(1 - \varepsilon/3)p|A||B| \leq e(A, B) \leq (1 + \varepsilon/3)p|A||B|.$$

We will show that, for sufficiently large n , G satisfies the property in the theorem. Let H be a $(1/2 + \varepsilon)$ -residual subgraph of G , with $|H| \geq \varepsilon n$, which contains a spanning $(2, C/p)$ -expander H_1 with $e(H_1) \leq \delta pn^2$. By **Q2**, for all subsets $A \subset V(H)$, $B \subset V(H) \setminus A$ with $|A| \geq C/p$ and $|B| \geq |H|/2$, we have

$$\begin{aligned} e_H(A, B) &\geq e_G(A, B) - (1/2 - \varepsilon)e_G(A, V(H)) \geq (1 - \varepsilon/3)p|A||B| - (1/2 - \varepsilon)(1 + \varepsilon/3)|A||H| \\ &\geq (1 - \varepsilon/3)p|A||B| - (1 - \varepsilon)p|A||B| = 2\varepsilon p|A||B|/3 \geq C\varepsilon^2 n/3. \end{aligned} \quad (3)$$

Let $H_2 \subset H$ be a random subgraph of H where each edge is included with probability $\delta/2$. Therefore, the probability there exists some disjoint $A, B \subset V(H)$ for which $|A| \geq C/p$, $|B| \geq |H|/2$ and $e_{H_2}(A, B) = 0$ is, by (3), at most

$$2^n \cdot 2^n \cdot (1 - \delta/2)^{C\varepsilon^2 n/3} \leq 4^n e^{-C\varepsilon^2 \delta n/6} = o(1).$$

By **Q2**, $e(G) \leq pn^2$, and thus, by Lemma 2.2, H_2 almost surely has at most δpn^2 edges. Therefore, we may take some $H_2 \subset H$ such that $e(H_2) \leq \delta pn^2$ and, for each disjoint $A, B \subset V(H)$ with $|A| \geq C/p$ and $|B| \geq |H|/2$, we have $e_{H_2}(A, B) > 0$. Let $H_0 = H_1 \cup H_2 \subset H$, noting that $e(H_0) \leq 2\delta pn^2$. We will show that H_0 is a 2-expander, so that, by **Q1**, H is Hamiltonian, as required.

Firstly, for each $A \subset V(H)$ with $C/p \leq |A| \leq |H|/8$, there are no edges between A and $V(H) \setminus (A \cup N_{H_2}(A))$ in H_2 , and, therefore, $|A \cup N_{H_2}(A)| \geq |H|/2$. Thus, $|N_{H_2}(A)| \geq |H|/2 - |A| \geq 3|H|/8 \geq 2|A|$.

Secondly, the graph H_0 is connected. Indeed, suppose to the contrary that H_0 has a component with vertex set A for which $|A| \leq |H|/2$. Then, as H_1 is a $(2, C/p)$ -expander, from $N_{H_0}(A) = \emptyset$ we have $|A| > C/p$. As there are no edges between A and $V(H) \setminus A$ in H_2 , this gives a contradiction.

As H_1 , and thus H_0 , is a $(2, C/p)$ -expander, H_0 is a 2-expander, as required. \square

5 Born resilience of Hamiltonicity

By Lemma 2.1 (with $p = M/\binom{n}{2}$) and Theorem 4.2, to prove Theorem 1.2 it is sufficient to show the following lemma.

Lemma 5.1. *For each $\delta, C > 0$, in almost every random graph process $\{G_M\}_{M \geq 0}$ with n vertices, for each M and $\varepsilon > 0$, if $\delta(G_M) \geq 2$, then any spanning $(1/2 + \varepsilon)$ -resilient subgraph $H \subset G_M$ contains a spanning $(2, C \binom{n}{2}/M)$ -expander with at most δM edges.*

Typically, in graphs in $\{G_M\}_{M \geq 0}$, sets of vertices with small degree (here, $\leq M/10^3 n$) will resiliently expand as these vertices form an independent set (see Section 5.1). Typically, sets with vertices with larger degree must resiliently expand to prevent overly dense subgraphs (see Section 5.2). Combining this, we prove Lemma 5.1 in Section 5.3.

5.1 Small degree vertices in the random graph process

As is well-known, in almost every random graph process $\{G_M\}_{M \geq 0}$ with n vertices, the first graph with minimum degree at least 2 has around $n(\log n + \log \log n)/2$ edges, as follows (see, for example, Bollobás [7, Theorems 2.2(ii) and 3.5]).

Lemma 5.2. *If $M = n(\log n + \log \log n - \omega(1))/2$, then, almost surely, $\delta(G_{n,M}) \leq 1$. If $k \in \mathbb{N}$ is fixed, and $M = n(\log n + (k-1)\log \log n + \omega(1))/2$, then, almost surely, $\delta(G_{n,M}) \geq k$.*

Furthermore, in almost every n -vertex random graph process $\{G_M\}_{M \geq 0}$, if $M \geq 25n \log n$, then there are no vertices with small degree in G_M .

Lemma 5.3. *In almost every random graph process $\{G_M\}_{M \geq 0}$ with n vertices, if $M \geq 25n \log n$, then $\delta(G_M) \geq M/n$.*

Proof. For each $M \geq 25 \log n$, let $p_M = M/\binom{n}{2}$ and $G = G(n, p)$. For each $v \in V(G)$, $\mathbb{E}(d_G(v)) = (n-1)p_M = 2M/n \geq 50 \log n$, so that, by Lemma 2.2,

$$\mathbb{P}(d_G(v) \leq M/n) \leq 2 \exp(-50 \log n/12) = o(n^{-4}).$$

Therefore, by Lemma 2.1, in almost every n -vertex random graph process $\{G_M\}_{M \geq 0}$, in every G_M with $M \geq 25n \log n$, we have $\delta(G_M) \geq M/n$. \square

As shown by Bollobás [6], in almost every random graph process, when the graph first has minimum degree 2 there are no vertices with low degree which are close together (see Lemma 5.4). We wish to show that, in almost every random graph process, such a property holds for *every* graph with minimum degree at least 2. Therefore, we repeat the argument in [6] to record some more detail.

Lemma 5.4. *Fix an integer $k > 1$, and let $n, M \in \mathbb{N}$. If $19n \log n/40 \leq M \leq 27n \log n$, then, with probability $1 - o(n^{-1/4})$, no two vertices with degree at most $\log n/36$ are within distance k of each other in the random graph $G = G_{n,M}$.*

Proof. Let $N = \binom{n}{2}$ and $d = \log n/36$. For each $a, b \in \mathbb{N}$ with $a \leq b$, use $(b)_a$ for the falling factorial $b(b-1)\cdots(b-a+1)$, and recall the simple result that, if $0 \leq a \leq b \leq c$, then $(b)_a/(c)_a \leq (b/c)^a$. Note also that, for sufficiently large n ,

$$\frac{Mn}{N-2n} = \frac{2M}{n-5} \leq 60 \log n \quad \text{and} \quad (2n-3d)M/N \geq 1.8 \log n. \quad (4)$$

The expected number of paths in $G = G_{n,M}$ with some length $1 \leq i \leq k$ whose endpoints together

have degree at most some $j \leq 2d$ is at most

$$\begin{aligned}
\sum_{i=1}^k n^{i+1} \sum_{j=1}^{2d} \binom{2n}{j} \binom{N-2n+3}{M-j-i} / \binom{N}{M} &= \sum_{i=1}^k n^{i+1} \sum_{j=1}^{2d} \binom{2n}{j} \cdot \frac{(N-M)_{2n-3-i-j}}{(N)_{2n-3-i-j}} \cdot \frac{(M)_{j+i}}{(N-2n+3+j+i)_{j+i}} \\
&\leq \sum_{i=1}^k n^{i+1} \sum_{j=1}^{2d} \binom{2n}{j} \cdot \left(1 - \frac{M}{N}\right)^{2n-3-j-i} \cdot \left(\frac{M}{N-2n}\right)^{j+i} \\
&\leq 2dk \cdot n^{k+1} \cdot \left(\frac{en}{d}\right)^{2d} \cdot e^{-(2n-3d)M/N} \cdot \left(\frac{M}{N-2n}\right)^{2d+k} \\
&\leq 2dkn \cdot \left(\frac{Mn}{N-2n}\right)^k \cdot \left(\frac{eMn}{d(N-2n)}\right)^{2d} \cdot e^{-(2n-3d)M/N} \\
&\stackrel{(4)}{\leq} 2dkn \cdot (60 \log n)^k \cdot \left(\frac{60e \log n}{d}\right)^{2d} \cdot e^{-1.8 \log n} \\
&\leq 2dk \cdot (60 \log n)^k \cdot e^{18d} \cdot n^{-0.8} \\
&\leq 2dk \cdot (60 \log n)^k \cdot e^{\log n/2} \cdot n^{-0.8} = o(n^{-1/4}).
\end{aligned}$$

Thus, with probability $1 - o(n^{-1/4})$, no two vertices with degree at most d are within distance k of each other in G . \square

The property in Lemma 5.4 is non-monotone, so we need slightly more work to show that it holds throughout almost every random graph process.

Lemma 5.5. *Let $k > 1$ be fixed. In almost every random graph process $\{G_M\}_{M \geq 0}$ with n vertices, if $19n \log n/40 \leq M \leq 25n \log n$, then there are no two vertices of degree at most $\log n/40$ within a distance k of each other in G_M .*

Proof. Let $N = \binom{n}{2}$, $d = \log n/36$ and $M_0 = n^{7/8}$. Letting $p = M_0/N \leq 3n^{-9/8}$, the probability the random graph $G(n, p)$ has a vertex with degree at least $d/10$ is at most

$$n \binom{n}{d/10} p^{d/10} \leq n \left(\frac{10enp}{d}\right)^{d/10} \leq n \left(\frac{30e}{dn^{1/8}}\right)^{d/10} = o(n^{1-d/100}) = o(n^{-3}).$$

From this, and Lemma 2.1, we have that, in almost every graph process $\{G_M\}_{M \geq 0}$ with n vertices, for each j , $0 \leq j \leq N - M_0$, $\Delta(G_{j+M_0} - G_j) \leq d/10$.

There are at most $25n^{1/8} \log n$ values of $j \in \mathbb{N}$ for which $19n \log n/40 \leq jM_0 \leq 25n \log n + M_0$. By Lemma 5.4 and a union bound, almost surely, for each such j , G_{jM_0} contains no two vertices with degree at most d within a distance k of each other. Now, suppose for some M , $19n \log n/40 \leq M \leq 25n \log n$, G_M has two vertices x and y with degree at most $\log n/40$ which are at most distance k apart in G_M . Find the smallest j for which $jM_0 \geq M$. Then $19n \log n/40 \leq jM_0 \leq 25n \log n + M_0$, but in G_{jM_0} both x and y have degree at most $\log n/40 + d/10 = d$ and are still within a distance k of each other in G_{jM_0} , a contradiction. Therefore, the property in the lemma almost surely holds for all M with $19n \log n/40 \leq M \leq 25n \log n$. \square

Corollary 5.6. *In almost every random graph process $\{G_M\}_{M \geq 0}$ with n vertices, in each G_M with $M \geq 19n \log n/40$ there are no two vertices with degree at most $M/10^3 n$ within a distance 5 of each other.*

Proof. Almost surely, by Lemma 5.5, in each G_M with $19n \log n/40 \leq M \leq 25n \log n$ there are no two vertices with degree less than $\log n/40 \geq M/10^3 n$ within a distance 5 of each other. Almost surely, by Lemma 5.3, for each G_M with $M \geq 25n \log n$, we have $\delta(G_M) > M/10^3 n$, so no vertices with degree at most $M/10^3 n$ exist. \square

5.2 Expansion from minimum degree conditions

Minimum degree conditions occur naturally in our $(1/2 + \varepsilon)$ -residual subgraphs in Lemma 5.1. The following lemma will allow us to convert these conditions into expansion properties.

Lemma 5.7. *For each $\delta > 0$, there exists $C > 0$ such that the following holds for $p \geq C/n$. With probability $1 - o(n^{-3})$, in the random graph $G = G(n, p)$ there are no two sets $A, B \subset V(G)$ with $|A| \leq 1/\delta p$, $|B| \leq 10|A|$ and $e(A, B) \geq \delta p n |A|$.*

Proof. For each value of t , with $1 \leq t \leq 1/\delta p$, if p_t is the probability that there are two sets $A, B \subset V(G)$ with $|A| = t$, $|B| = 10t$ and $e(A, B) \geq \delta p n t$, then, for sufficiently large C ,

$$\begin{aligned} p_t &\leq \binom{n}{t} \binom{n}{10t} \binom{10t^2}{\delta p n t / 2} p^{\delta p n t / 2} \leq \left(\frac{en}{t}\right)^{11t} \left(\frac{20et}{\delta n}\right)^{\delta p n t / 2} \leq \left(\frac{20e^2}{\delta}\right)^{11t} \left(\frac{20et}{\delta n}\right)^{\delta C t / 2 - 11t} \\ &\leq \left(\frac{400e^3 t}{\delta^2 n}\right)^{11t}. \end{aligned}$$

Therefore, if $1 \leq t < \log n$, then, $p_t = o((1/\sqrt{n})^{11}) = o(n^{-4})$. If $\log n \leq t \leq 1/\delta p$, then $p_t \leq (400e^3/\delta^3 p n)^{11 \log n} \leq (400e^3/\delta^3 C)^{11 \log n} = o(n^{-4})$, for sufficiently large C . Therefore, no two such sets exist for any $t \leq 1/\delta p$ with probability $1 - o(n^{-3})$. \square

5.3 Resilient small set expansion

We can now combine the work in Sections 5.1 and 5.2 to prove Lemma 5.1.

Proof of Lemma 5.1. Note that we can assume that $\delta \leq 10^{-4}$ and $C \geq 16/\delta$. Almost surely, by Lemma 5.2, for each G_M with $M \leq 19n \log n / 40$ we have $\delta(G_M) < 2$, so we need only consider the range $M \geq 19 \log n / 40$. By Corollary 5.6, for each $M \geq 19 \log n / 40$, there are no vertices with degree at most $M/10^3 n$ within distance 5 of each other.

By Lemmas 2.1 and 5.7, we almost surely get the following property for each graph G_M , $M \geq 19n \log n / 40$, with $p = M/\binom{n}{2}$.

R There are no two sets $A, B \subset V(G_M)$ with $|A| \leq C/p$, $|B| \leq 10|A|$ and $e(A, B) \geq \delta p n |A|/8$.

The random graph process then has the property in the lemma, as follows.

Fixing $M \geq 19 \log n / 40$, $p = M/\binom{n}{2}$ and $\varepsilon > 0$, suppose $\delta(G_M) \geq 2$ and let H be a $(1/2 + \varepsilon)$ -residual spanning subgraph of G_M . Note that $\delta(H) \geq 2$. For each $v \in V(H)$, take $\min\{d_H(v), \delta p n / 4\}$ edges incident to v , and use these edges to create a new graph H_0 with $e(H_0) \leq \delta p n^2 / 4 \leq \delta M$ and $d_{H_0}(v) \geq \min\{d_H(v), \delta p n / 4\} \geq 2$ for each $v \in V(H)$.

Let $A \subset V(H)$ with $|A| \leq C/p$. Let $A_1 \subset A$ be the set of vertices not within distance 2 of another vertex in A in H_0 , and let $A_2 = A \setminus A_1$. As $\delta(H_0) \geq 2$, $|N_{H_0}(A)| \geq 2|A_1| + |N_{H_0}(A_2)|$. Now, two vertices in A with degree at most $M/10^3 n$ in G_M cannot be within distance 2 of the same vertex in A in H_0 or each other. Therefore, at least $|A_2|/2$ vertices in A_2 must have degree at least $M/10^3 n$ in G_M , and hence degree $\delta p n / 4$ in H_0 . Thus, $e(A_2, A_2 \cup N_{H_0}(A_2)) \geq \delta p n |A_2|/8$, so that, by **R**, $|N_{H_0}(A_2)| \geq 2|A_2|$. Therefore, $|N_{H_0}(A)| \geq 2|A_1| + |N_{H_0}(A_2)| \geq 2|A|$. Hence, as required, $H_0 \subset H$ is a $(2, C/p)$ -expander with $e(H_0) \leq \delta M$ which spans H . \square

6 Born resilience of Hamiltonicity in the k -core

Our key result in this section is the following lemma, showing that the k -core of $G(n, p)$ almost always resiliently contains a sparse spanning $(2, \Theta(1/p))$ -expander.

Lemma 6.1. *For each $\delta, C > 0$, there exists k_0 such that, for each $k \geq k_0$, $n \in \mathbb{N}$ and $p > 0$ with $k/n \leq p \leq 2 \log n/n$, the following holds. With probability $1 - o(n^{-3})$, $G = G(n, p)$ has the property that any spanning $1/2$ -residual subgraph of $G^{(k)}$ contains a spanning $(2, C/p)$ -expander with at most $\delta p n^2$ edges.*

Following Krivelevich, Lubetzky and Sudakov [15], we will consider two cases: the *critical case* where $p \leq 4k/n$ (in Section 6.2) and the *supercritical case* where $p \geq 4k/n$ (in Section 6.3). We use a lemma from [15], but note that its proof relies on a relatively simple calculation and that our methods for demonstrating Hamiltonicity in the k -core (for large k) are quite different from those used by Krivelevich, Lubetzky and Sudakov in [15].

We will show first, in Section 6.1, that Theorem 1.3 follows from Lemma 6.1, Theorem 4.2 and some standard results on the likely size and appearance of the k -core.

6.1 Typical properties of the k -core

Luczak [17] showed that, when the k -core first appears in the random graph process it is likely to be linear in size, as follows.

Theorem 6.2 ([17]). *For each $k \geq 3$, in almost every n -vertex random graph process $\{G_M\}_{M \geq 0}$, for each M , either $G^{(k)} = \emptyset$ or $|G^{(k)}| \geq n/5000$. \square*

The threshold of appearance for the k -core has been well-studied (see [18, 19, 21]), showing that, as the first $kn/2$ edges are added in the n -vertex random graph process, the k -core is likely to be empty, as follows (see also [15]).

Theorem 6.3. *For each $k \geq 20$, in almost every n -vertex random graph process $\{G_M\}_{M \geq 0}$, if $M \leq kn/2$, then $G_M^{(k)} = \emptyset$.*

Theorem 1.3 is implied by Lemma 6.1 and Theorems 4.2, 6.2, and 6.3, as follows.

Proof of Theorem 1.3. Using Theorem 4.2, let $\delta, C > 0$ be such that, if $p \geq C/n$, then $G = G(n, p)$ has the following property with probability $1 - o(n^{-3})$.

S1 If H is a $(1/2 + \varepsilon)$ -residual subgraph of G with $|H| \geq n/5000$ which contains a spanning $(2, C/p)$ -expander with at most $\delta p n^2$ edges, then H is Hamiltonian.

Using Lemma 6.1, let $k_0 \geq C$ be such that, for each $k \geq k_0$ and $k/n \leq p \leq 2 \log n/n$, $G = G(n, p)$ has the following property with probability $1 - o(n^{-3})$.

S2 If $G^{(k)} \neq \emptyset$, then any spanning $1/2$ -residual subgraph of $G^{(k)}$ contains a spanning $(2, C/p)$ -expander with at most $\delta p n^2$ edges.

Increase k_0 if necessary, so that, using Lemmas 2.1 and 1.7, we have the following property for almost every n -vertex random graph process $\{G_M\}_{M \geq 0}$.

S3 For each $M \geq k_0 n/2$, there is no $U \subset V(G_M)$ with $|U| \geq n/5000$ which is $(1/2 + \varepsilon)$ -resiliently Hamiltonian.

Now, let $k \geq k_0$, and consider the random graph process $\{G_M\}_{M \geq 0}$ with n vertices. By Lemma 5.2, $\delta(G_M)$ is almost surely at least k for each $M \geq (n-1) \log n$, in which case $G_M^{(k)} = G_M$. Thus, almost surely, by Theorem 1.2, for each $M \geq (n-1) \log n$, G_M is $(1/2 - \varepsilon)$ -resiliently Hamiltonian but not

$(1/2+\varepsilon)$ -resiliently Hamiltonian. Furthermore, by Lemma 6.3, almost surely, if $M \leq kn/2$, then $G_M^{(k)} = \emptyset$. Therefore, we can assume that $kn/2 \leq M \leq (n-1) \log n$.

Almost surely, **S3** holds and, for each $kn/2 \leq M \leq (n-1) \log n$, G_M satisfies **S1** and **S2** with $p = M/\binom{n}{2}$, so that $k \leq pn \leq 2 \log n$. Almost surely, by Theorem 6.2, for each $kn/2 \leq M \leq (n-1) \log n$, $G_M^{(k)} = \emptyset$ or $|G_M^{(k)}| \geq n/5000$. Then, if $G_M^{(k)} \neq \emptyset$, by **S1** and **S2**, $G_M^{(k)}$ is $(1/2 - \varepsilon)$ -resiliently Hamiltonian, but, by **S3**, not $(1/2 + \varepsilon)$ -resiliently Hamiltonian. \square

6.2 Critical k -core expansion

We can now prove Lemma 6.1 in the critical case.

Proof of Lemma 6.1 when $p \leq 4k/n$. Note that we can assume that $\delta \leq 1/8$. Let k_0 be sufficiently large that, by Lemma 5.7, the following holds in $G(n, p)$ with $p \geq k_0/n$ with probability $1 - o(n^{-3})$.

T There are no two sets $A, B \subset V(G)$ with $|A| \leq C/p$, $|B| \leq 10|A|$ and $e(A, B) \geq \delta pn|A|/2$.

Let $k \geq k_0$ and $k/n \leq p \leq 4k/n$. Let $G = G(n, p)$, so that, with probability $1 - o(n^{-3})$, G satisfies **T**.

Let H be a spanning $1/2$ -residual subgraph of $G^{(k)}$. Note that $\delta pn \leq \delta(4k) \leq k/2$, so that $\delta(H) \geq k/2 \geq \delta pn/2$. For each vertex $v \in V(H)$ take $\delta pn/2$ edges incident to v , and use these edges to form a subgraph $H_0 \subset H$ with $e(H_0) \leq \delta pn^2$. By **T**, for each $A \subset V(H_0)$ with $|A| \leq C/p$, we have $|N_{H_0}(A) \cup A| \geq 10|A|$, and hence H_0 is a $(2, C/p)$ -expander. \square

6.3 Supercritical k -core expansion

For the resilient expansion of small sets in the k -core, we will use the following lemma taken from a result of Krivelevich, Lubetzky and Sudakov [15, Claim 3.1].

Lemma 6.4. *Let $p = c/n$ with $1 < c < \log^2 n$ and let $G = G(n, p)$. With probability $1 - o(n^{-3})$, any subgraph $H \subset G$ with $\delta(H) \geq 15$ is a $(2, n/2c^2)$ -expander.* \square

To show that medium-sized sets resiliently expand in the k -core of a typical random graph $G(n, p)$ with $p \geq 4k/n$, we wish to use Lemma 5.7. To do so, we will show that it is very likely that most of the vertices in the k -core here have at least $pn/4$ neighbours in the k -core, as follows.

Lemma 6.5. *There exists $k_0 \geq 0$ such that the following holds for any $k \geq k_0$ and $p = c/n$ with $4k \leq c \leq 2 \log n$. With probability $1 - o(n^{-3})$, in $G = G(n, p)$ there are at most $n/4c^2$ vertices with degree at most $pn/4$ in $G^{(k)}$.*

Proof. For each $A \subset V(G)$ with $|A| = n/4c^2$, the expected number of edges between A and $V(G) \setminus A$ is

$$p|A||V(G) \setminus A| \geq 3p|A|n/4 = 3n/16c.$$

Therefore, the probability there is some set $A \subset V(G)$ with $|A| = n/4c^2$ and $e_G(A, V(G) \setminus A) < |A| \cdot pn/4$ is, using Proposition 2.3, at most

$$\binom{n}{n/4c^2} \exp(-3n/(9 \cdot 4 \cdot 16c)) \leq (4ec^2)^{n/4c^2} \exp(-n/200c) \leq \exp((3 + 2 \log c - c/50)n/4c^2) = o(n^{-3}),$$

where $c \geq 4k_0$ is sufficiently large.

Suppose then that G has no such set A . Remove vertices with degree at most $pn/4$ iteratively from G until no such vertices exist, or we have removed $n/4c^2$ vertices. Let A be the set of removed vertices. Then, $e_G(A, V(G) \setminus A) < |A| \cdot pn/4$, so that $|A| < n/4c^2$. Furthermore, $\delta(G - A) > pn/4 \geq k$. Thus, $G - A \subset G^{(k)}$ and $V(G) \setminus A$ is a set of at least $(1 - 1/4c^2)n$ vertices with degree more than $pn/4$ in $G^{(k)}$. \square

We can now prove Lemma 6.1 in the supercritical case.

Proof of Lemma 6.1 when $p \geq 4k/n$. Note that we can assume that $\delta < 1/8$. Using Lemmas 5.7 and 6.5, let $k_0 \geq 30/\delta$ be sufficiently large that, if $p \geq 4k_0/n$, then the following hold in $G = G(n, p)$ with probability $1 - o(n^{-3})$.

U1 There are no two sets $A, B \subset V(G)$ with $|A| \leq C/p$, $|B| \leq 10|A|$ and $e(A, B) \geq \delta pn|A|/4$.

U2 There are at most $n/2c^2$ vertices in $G^{(k)}$ with degree at most $pn/4$.

Assume that $G = G(n, p)$ has these properties. By Lemma 6.4, with probability $1 - o(n^{-3})$, G has the following property.

U2 If $H_0 \subset G$ has minimum degree 15, then it is a $(2, n/c^2)$ -expander.

Let H be a spanning $1/2$ -residual subgraph of $G^{(k)}$. Note that $\delta(H) \geq k/2 \geq 15$. For each vertex $v \in V(H)$, take $\min\{d_H(v), \delta pn/2\} \geq 15$ edges incident to v , and use these edges to form a subgraph $H_0 \subset H$ with $e(H_0) \leq \delta pn^2$.

Let $A \subset V(H_0)$ with $n/2c^2 \leq |A| \leq C/p$. By **U2**, there are at least $|A|/2$ vertices in A with degree at least $\delta pn/2$ in H_0 . Thus, by **U1**, we have $|N_{H_0}(A) \cup A| \geq 10|A|$, so that $|N_{H_0}(A)| \geq 2|A|$. Therefore, as $\delta(H_0) \geq 15$, by **U2**, H_0 is a $(2, C/p)$ -expander. \square

This completes the proof of Lemma 6.1, and thus Theorem 1.3 is proved.

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